2018

Geometry of Derived Categories on Noncommutative Projective Schemes

Blake Alexander Farman
University of South Carolina

Geometry of Derived Categories on Noncommutative Projective Schemes

by

Blake Alexander Farman

Bachelor of Science
Rensselaer Polytechnic Institute 2009

Master of Science
University of Vermont 2011

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2018

Accepted by:

Matthew R. Ballard, Major Professor

Frank Thorne, Committee Member

Jesse Kass, Committee Member

Andrew Kustin, Committee Member

Michael Filaseta, Committee Member

Karl Gregory, Committee Member

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

Dedication

To Professor Richard M. Foote, for all your generosity. Your appreciation for the beauty of mathematics taught me to love the subject, and your wonderful guidance taught me not only how to do it, but how to write it. In all the mathematics I have written, I have always written as though it could appear in a text book. I can only hope that this piece lives up to that standard.

Acknowledgments

I owe a great debt of gratitude to Richard Foote, David Dummit, and John Voight for their outsized impact at the beginning of this journey at the University of Vermont eight years ago. But for their outstanding classes, I do not believe I would be where I am now. I would also like to thank my dear friends Paul Lessard and John Willis, both of whom I was incredibly fortunate to meet as classmates at the University of Vermont, for helping make that experience, amongst a great many others, truly exceptional.

I am extremely grateful for the support and encouragement of the faculty at the University of South Carolina over the past six years, particularly Andy Kustin and Frank Thorne for all of your wonderful courses and helpful advice. Most of all, I would like to thank my advisor, Matt Ballard, for your very patient guidance, encouragement, and unwavering support.

I would also like to thank all of my friends and family for their support throughout this process. I would like to thank everyone who participated regularly in the Fluid Dynamics 'seminar,' through which I have met many of my friends here at the University of South Carolina, and my fellow captains who have helped navigate that wayward ship.

Finally, I would like to especially thank my beloved wife, Ann Clifton, for all your love, encouragement and support. I, too, look forward to a lifetime of love, joy, and mathematics with you.

Abstract

Noncommutative Projective Schemes were introduced by Michael Artin and J.J. Zhang in their 1994 paper of the same name as a generalization of projective schemes to the setting of not necessarily commutative algebras over a commutative ring. In this work, we study the derived category of quasi-coherent sheaves associated to a noncommutative projective scheme with a primary emphasis on the triangulated equivalences between two such categories.

We adapt Artin and Zhang's noncommutative projective schemes for the language of differential graded categories and work in $\mathrm{Ho}\,(\mathrm{dgcat}_k)$, the homotopy category of differential graded categories, making extensive use of Bertrand Toën's Derived Morita Theory. For two noncommutative projective schemes, X and Y, we associate differential graded enhancements, $\mathcal{D}(X)$ and $\mathcal{D}(Y)$, of the respective derived categories of quasi-coherent sheaves. Under appropriate cohomological conditions, we provide a noncommutative geometric description of the subcategory, $\mathbf{R}\underline{\mathrm{Hom}}_c(\mathcal{D}(X), \mathcal{D}(Y))$, of the internal Hom category in $\mathrm{Ho}\,(\mathrm{dgcat}_k)$. As an immediate application, we show that, under these conditions, any triangulated equivalence between the derived categories induces an equivalence of Fourier-Mukai type, with kernel an object of the derived category of quasi-coherent sheaves on the appropriate analogue of the product.

Preface

Derived Categories

Derived categories were initially conceived by Grothendieck as a device for maintaining cohomological data during his reformulation of algebraic geometry through scheme theory, and were fleshed out by his student, Verdier, in his thesis (Verdier 1996). While not immediately apparent, over time this object, originally devised as a sort of book keeping device, has been recognized as the key to linking algebraic geometry to a broad range of subjects both within and without mathematics. As such, the study of derived categories has risen to prominence as a central subfield of algebraic geometry. In particular, Bridgeland attributes this growth to three main applications in his 2006 ICM address (Bridgeland 2006).

The first is the deep interrelationship between algebraic geometry and string theory. In his 1994 ICM address (Kontsevich 1995), Kontsevich conjectures that dualities seen in string theory should be expressed mathematically as a derived equivalence between the Fukaya category and the category of coherent sheaves on a complex algebraic variety. In the ensuing years, homological mirror symmetry has grown into a mathematical subject in its own right. Indeed, the physical intuition which homological mirror symmetry seeks to harness has already led to fruitful study of enumerative problems in algebraic geometry (Candelas et al. 1991).

The second is the wealth of information maintained in the derived category which has been hidden away from even modern geometric approaches. Work of Mukai (1981); Mukai (1987) demonstrates that moduli spaces of sheaves on a variety can be encoded in the derived category. Work of Bondal and Orlov (1995) shows how one can attack birational geometry through the derived category, by encoding blow-ups, which are foundational objects of birational geometry, as semi-orthogonal decompositions.

Moreover, much work in the direction of derived categories in algebraic geometry have yielded fruitful classification results. Thanks to Orlov (1997), it is known that over an algebraically closed field, curves are derived equivalent if and only if they are isomorphic. In dimension two, for X smooth and projective, but not elliptic, K3, nor abelian, it is known that derived equivalence implies isomorphism (Huybrechts 2006, Prop. 12.1). In higher dimension, it was originally conjectured in Kawamata (2002) that there are only finitely many derived equivalent surfaces up to isomorphism. In Anel and Toën (2009) it was shown that there are at most countably many varieties in the derived equivalence class, while the original conjecture is shown to be false in Lesieutre (2015).

Of central importance in each of the situations above are the so-called kernels of Fourier-Mukai transforms. For smooth projective varieties, X and Y, the kernels are objects in the derived category of $X \times_k Y$ which induce an equivalence of their respective derived categories, this equivalence being called a Fourier-Mukai transform. The main theorem of Orlov (1997) is that equivalences of derived categories of smooth projective varieties arise from these kernels. The spectacular advantage of having kernels is the translation of an equivalence of derived categories, which is intrinsically cohomological data at the level of triangulated categories, to geometric data encoded by the kernel. The potency of this relationship is borne out by tying the minimal model program of birational geometry to semi-orthogonal decompositions of the derived category in Bridgeland (2002); Kawamata (2002) and the notion of Bridgeland stability in Bridgeland (2007); Arcara et al. (2013); Bayer and Macrì (2014a); Bayer and Macrì (2014b), which demonstrate the mixture of derived categories, moduli spaces, and birational geometry.

The final point, and the main topic of this work, is that the methods of derived categories may yet serve as the dictionary between the methods of projective algebraic geometry and the study of noncommutative algebra. While a direct generalization of schemes to noncommutative rings is, in some sense, highly pathological, one does have a good notion of quasi-coherent and coherent sheaves. The success in the commutative case to express geometric phenomena through the derived category of coherent sheaves suggests that the noncommutative analogue should serve as a bridge between these worlds.

Noncommutative Projective Schemes

The deep interrelationship between commutative algebra and algebraic geometry has been well known for quite some time. More recently, in an effort to understand the world of noncommutative algebra, Artin and Zhang (1994) introduced Noncommutative Projective Schemes as the noncommutative analogues of geometric objects associated to graded rings. This work stems largely from Artin and Schelter (1987) in which an attempt at classifying the noncommutative analogues of $\mathbb{P}^2$ was made.

In the commutative situation, one associates to a graded ring, A, the scheme $X = \operatorname{Proj} A$, the projective spectrum, along with the categories $\operatorname{Qcoh} X$ of quasi-coherent sheaves and $\operatorname{coh} X$ of coherent sheaves. Analogously, to a noncommutative graded algebra, A, over a commutative ring, k, one associates the category $\operatorname{QGr} A$, declared to be the category of quasi-coherent sheaves. This category is obtained as the quotient of the category, $\operatorname{Gr} A$, of graded modules by the Serre subcategory of torsion graded modules, $\operatorname{Tors} A$, in the sense of Gabriel (1962). While these schemes do not, in general, admit a space on which to do geometry, they do provide what are arguably the fundamental objects of study in modern algebraic geometry: the quasi-coherent sheaves and its full noetherian subcategory, $\operatorname{qgr} A$, of coherent sheaves. The precise justification for this definition rests on the following famous theorem of Serre: If A is a commutative graded ring generated in degree one, the category of quasi-coherent sheaves on Proj A is equivalent to the quotient category, QGrA, and the category of coherent sheaves on Proj A is equivalent to its full noetherian subcategory, qgr A.

Of late, much work has been done on the classification of noncommutative varieties of low dimension. The tools of birational geometry and moduli spaces from projective algebraic geometry have been adapted to this noncommutative projective algebraic geometry to great success. In dimension one, methods of noncommutative birational geometry account for the classification of all noncommutative curves which is due to Artin and Stafford (1995) and Reiten and Van den Bergh (2002). However, as indicated in Stafford's 2002 ICM address (Stafford 2002), the question of classifying noncommutative surfaces remains open. In Artin (1997), Artin conjectured that, up to birational equivalence, there are four types of surfaces. Towards this end, partial classification results for noncommutative surfaces have been given in Artin, Tate, and Van den Bergh (1990); Stephenson (1996); Stephenson (1997) using methods of moduli spaces.

The guiding principle set forth by Artin and Zhang is that our understanding of projective algebraic geometry should drive our intuition in the study of noncommutative algebra. Indeed, the recent results above have been largely due to adaptations of some of these methods and, given the significant advances in the commutative setting, one should expect that derived categories will play a leading role in this study. However, conspicuously absent from this accounting are any such developments. As was the case in the commutative setting, the primary stumbling block appears in large part to be the absence of Fourier-Mukai kernels. Having such a statement for the case of noncommutative projective schemes therefore seems of high priority.

Table of Contents

Chapter 1

Introduction

Fourier-Mukai Kernels for Noncommutative Projective Schemes

In light of their absense in noncommutative projective geometry, the natural question to ask is what these kernels should be. Toën's derived Morita theory (Toën 2007) gives an overarching framework to attack such a problem by abstracting to the higher categorical structure of differential graded (dg) categories. Working within the homotopy category of the 2-category of all small dg-categories over a commutative ring, Toën is able to provide an incredibly elegant reformulation of Fourier-Mukai functors at the level of pre-triangulated dg-categories via the dg-subcategory, $\mathbf{R}\underline{\mathrm{Hom}}_c$, of the internal Hom. Indeed, using this machinery, kernels have been recovered for schemes in Toën (2007), and obtained for higher derived stacks in Ben-Zvi, Francis, and Nadler (2010) and for categories of matrix factorizations in Dyckerhoff (2011); Polishchuk and Vaintrob (2012); Ballard, Favero, and Katzarkov (2014). In each case, the work lies in the identification of the internal Hom object obtained from this machinery within the theory from which the input dg-categories originate, for even if they arise geometrically, the resulting Hom is often quite abstract.

The obvious first step in such work is to identify the possible input dg-categories for the machinery of derived Morita theory. In the situation of interest, one considers the noncommutative projective scheme, $\mathrm{QGr}\, A$, associated to the connected graded algebra, A, over a field, k. The natural choice of dg-category is the dg-enhancement, $\mathcal{D}(\mathrm{QGr}\, A)$, of the derived category $\mathrm{D}(\mathrm{QGr}\, A)$, in the sense of Lunts and Orlov (2010), which is unique up to equivalence in the homotopy category of dg-categories. One must then identify the dg-category $\mathbf{R}\underline{\mathrm{Hom}}_c(\mathcal{D}(\mathrm{QGr}\, A), \mathcal{D}(\mathrm{QGr}\, B))$ noncommutative geometrically.

Generally, care must be taken to ensure good behavior of $\mathrm{QGr}\, A$, but one may exert some control by imposing cohomological conditions on the ring, A. Two such common conditions are the Ext-finite condition of Bondal and Van den Bergh (2003) and the condition $\chi^\circ(M)$ of Artin and Zhang (1994). One can interpret these conditions geometrically as imposing Serre vanishing for the noncommutative twisting sheaves together with a local finite dimensionality over the ground field, k. Specifically, one can force good behavior with respect to Toën's derived Morita theory by requiring that two connected graded algebras, A and B, over a field, k, are both left and right Noetherian, Ext-finite, and satisfy the condition $\chi^\circ(M)$ for the left/right A-modules $M = A, A^{\mathrm{op}}$, and the left/right B-modules $M = B, B^{\mathrm{op}}$. We call such a pair of algebras a *delightful couple*.

In this work we establish the identification

$$\mathbf{R}\underline{\mathrm{Hom}}_c(\mathcal{D}(\mathrm{QGr}\, A), \mathcal{D}(\mathrm{QGr}\, B)) \cong \mathcal{D}(\mathrm{QGr}(A^{\mathrm{op}} \otimes_k B))$$

in the homotopy category of dg-categories under these hypotheses.

As an easy corollary of the main result, one has the following statement.

Theorem 1.0.1. *Let X and Y be noncommutative projective schemes associated to a delightful couple over a field k, both of which are generated in degree one. Then for any equivalence $\mathrm{D}(\mathrm{Qcoh}\, X) \to \mathrm{D}(\mathrm{Qcoh}\, Y)$, there exists an object P of $\mathrm{D}(\mathrm{Qcoh}\, X \times_k Y)$ whose associated integral transform is an equivalence of Fourier-Mukai type.*

The interested reader can see Corollary 5.4.3 for a more careful statement of this result.

Conventions

The ring k will always be at least Noetherian and commutative, though often will be a field. Often, for ease of notation, $\mathcal{C}(X,Y)$ will be used to refer to the morphims, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, between objects X and Y of a category $\mathcal{C}$, though we shall also use an undecorated Hom depending on the complexity of the notation, provided the meaning is clear from context. Whenever $\mathcal{C}$ has a natural enrichment over a category, $\mathcal{V}$, we will denote by $\underline{\mathcal{C}}(X,Y)$ the $\mathcal{V}$-object of morphisms.

For example, the category of complexes of k-vector spaces, $C\left(k\right)$, can be endowed with the the structure of a $C\left(k\right)$-enriched category using the hom total complex,

$$\mathcal{C}\left(k\right)\left(C,D\right) := \underline{C}(k)(C,D)$$

which has in degree n the k-vector space

$$\mathcal{C}\left(k\right)\left(C,D\right)^n = \prod_{m\in\mathbb{Z}} \mathrm{Mod}\, k\left(C^m, D^{m+n}\right)$$

and differential

$$d(f) = d_D \circ f + (-1)^{n+1} f \circ d_C.$$

It should be noted that $Z^0(\mathcal{C}\left(k\right)\left(C,D\right)) = C\left(k\right)\left(C,D\right)$.

Chapter 2

Differential Graded Categories

In this chapter we recall some basic facts about differential graded (dg) categories. For a more detailed treatment of dg-categories, see, e.g., Keller (1994); Keller (2006); Drinfeld (2004). For a detailed treatment of enriched categories, see, e.g., Borceux (1994, Chapter 6).

Recall that a **dg-category**, $\mathcal{A}$, over k is a category enriched over the category of chain complexes, $C(k)$, a **dg-functor**, $F\colon \mathcal{A} \to \mathcal{B}$ is a $C(k)$-enriched functor, a **morphism of dg-functors of degree** n, $\eta\colon F \to G$, is a $C(k)$-enriched natural transformation such that $\eta(A) \in \mathcal{B}(FA, GA)^n$ for all objects A of $\mathcal{A}$, and a **morphism of dg-functors** is a degree zero, closed morphism of dg-functors. We will denote by dgcat_k the 2-category of small $C(k)$-enriched categories, and by $\underline{\mathrm{dgcat}}_k(\mathcal{A}, \mathcal{B})$ the dg-category of dg-functors from $\mathcal{A}$ to $\mathcal{B}$.

Recall also that for $\mathcal{A}$ and $\mathcal{B}$ small dg categories, we may define a dg-category $\mathcal{A} \otimes \mathcal{B}$ with objects $\mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{B})$ and morphisms

$$(\mathcal{A} \otimes \mathcal{B})\left((X, Y), (X', Y')\right) = \mathcal{A}(X, X') \otimes_k \mathcal{B}(Y, Y').$$

It is well known that there is an isomorphism

$$\mathrm{dgcat}_k(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{dgcat}_k(\mathcal{A}, \underline{\mathrm{dgcat}}_k(\mathcal{B}, \mathcal{C})),$$

endowing dgcat_k with the structure of a symmetric monoidal closed category.

For any dg-category, $\mathcal{A}$, we denote by $Z^0(\mathcal{A})$ the category with objects those of $\mathcal{A}$ and morphisms

$$Z^0(\mathcal{A})(A_1, A_2) := Z^0(\mathcal{A}(A_1, A_2)).$$

By $H^0(\mathcal{A})$ we denote the category with objects those of $\mathcal{A}$ and morphisms

$$H^0(\mathcal{A})(A_1, A_2) := H^0(\mathcal{A}(A_1, A_2)).$$

Following Canonaco and Stellari (2015), we say that two objects A_1, A_2 of a dg-category, $\mathcal{A}$, are **dg-isomorphic** (respectively, **homotopy equivalent**) if there is a morphism $f \in Z^0(\mathcal{A})(A_1, A_2)$ such that f (respectively, the image of f in $H^0(\mathcal{A})(A_1, A_2)$) is an isomorphism. In such a case, we say that f is a **dg-isomorphism** (respectively, **homotopy equivalence**).

2.1 The Model Structure on DG-Categories

We collect here some basic results on the model structure for dgcat_k. Our standard reference for model categories in general is Hovey (1999).

For any dg-functor $F\colon \mathcal{A} \to \mathcal{B}$, we say that F is

(i) **quasi-fully faithful** if for any two objects A_1, A_2 of $\mathcal{A}$ the morphism

$$F(A_1, A_2)\colon \mathcal{A}(A_1, \mathcal{A}_2) \to \mathcal{B}(FA_1, FA_2)$$

is a quasi-isomorphism of chain complexes,

(ii) **quasi-essentially surjective** if the induced functor $H^0(F)\colon H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is essentially surjective,

(iii) a **quasi-equivalence** if F is quasi-fully faithful and quasi-essentially surjective,

(iv) a **fibration** if F satisfies the following two conditions:

(a) for all objects A_1, A_2 of $\mathcal{A}$, the morphism $F(A_1, A_2)$ is a degree-wise surjective morphism of complexes, and

(b) for any object A of $\mathcal{A}$ and any isomorphism $\eta \in H^0(\mathcal{B})(H^0(F)A, B)$, there exists an isomorphism $\nu \in H^0(\mathcal{C})(A, A')$ such that $H^0(F)(\nu) = \eta$.

In Tabuada (2005) it is shown that taking the class of fibrations defined above and the class of weak equivalences to be the quasi-equivalences, dgcat_k becomes a cofibrantly generated model category. The localization of dgcat_k at the class of quasi-equivalences is the homotopy category, $\mathrm{Ho}\,(\mathrm{dgcat}_k)$. We will denote by $[\mathcal{A}, \mathcal{B}]$ the morphisms of $\mathrm{Ho}\,(\mathrm{dgcat}_k)$.

A small dg-category $\mathcal{A}$ is said to be **h-projective** if for all objects A_1, A_2 of $\mathcal{A}$ and any acyclic complex, C, every morphism of complexes $\mathcal{A}(A_1, A_2) \to C$ is null-homotopic. In Canonaco and Stellari (2015), it is shown that there exists an h-projective category, $\mathcal{A}^{\mathrm{hp}}$, quasi-equivalent to $\mathcal{A}$ and, as a result, the localization of the full subcategory of dgcat_k of h-projective dg-categories at the class of quasi-equivalences is equivalent to $\mathrm{Ho}\,(\mathrm{dgcat}_k)$. In particular, when k is a field, every dg-category is h-projective and hence one can compute the derived tensor product by

$$\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B} = \mathcal{A}^{\mathrm{hp}} \otimes \mathcal{B} = \mathcal{A} \otimes \mathcal{B}.$$

We will make extensive use of this fact throughout.

2.2 Differential Graded Modules

Before making the relevant definitions, we pause for a brief justification of the use of the word module. To a ring A, one can associate the Ab-enriched category, $\mathcal{A}$, with one object, endomorphisms the abelian group A, and composition given by multiplication. We will refer to the category $\mathcal{A}$ as the **ringoid associated to** A. As one is wont to do in mathematics, we shift perspective by invoking enriched category theory and abstract away to the 2-category, Ab-cat, of all small Ab-enriched categories. Indeed, it is an easy exercise in translation that one recovers the classical category of A-modules as the Ab-enriched category of Ab-enriched functors, $\text{Ab-cat}(\mathcal{A}, \mathrm{Ab})$.

More generally, for any Ab-enriched category, $\mathcal{A}$, one could reasonably call $\text{Ab-cat}(\mathcal{A}, \mathrm{Ab})$ the category of Ab-modules over $\mathcal{A}$; the classical A-modules could then be regarded as Ab-modules over the ringoid $\mathcal{A}$. Since these constructions really only rely on the fact that Ab is a symmetric monoidal closed category, one is naturally led to think about mimicking this construction with another category, $\mathcal{V}$, of the same type. This of course leads to $\mathcal{V}$-modules over a $\mathcal{V}$-category, $\mathcal{A}$. As dg-categories are just $\mathcal{V}$-enriched categories for $\mathcal{V} = C(k)$, we adopt the name dg-module.

For any small dg-category, $\mathcal{A}$, denote

$$\mathrm{dgMod}(\mathcal{A}) := \underline{\mathrm{dgcat}}_k(\mathcal{A}^{\mathrm{op}}, \mathcal{C}(k)),$$

the dg-category of dg-functors, where $\mathcal{C}(k)$ denotes the dg-category of chain complexes equipped with the internal Hom from its symmetric monoidal closed structure. The objects of $\mathrm{dgMod}(\mathcal{A})$ will be called **dg $\mathcal{A}$-modules**. Since one may view the dg $\mathcal{A}^{\mathrm{op}}$-modules as what should reasonably be called left dg $\mathcal{A}$-modules, the terms right and left will be dropped in favor of dg $\mathcal{A}$-modules and dg $\mathcal{A}^{\mathrm{op}}$-modules, respectively. We note here that the somewhat vexing choice of terminology is such that we can view objects of $\mathcal{A}$ as dg $\mathcal{A}$-modules by way of the enriched Yoneda embedding

$$Y_{\mathcal{A}} \colon \mathcal{A} \to \mathrm{dgMod}(\mathcal{A}).$$

Just as one usually calls an abelian group with compatible left A-action and right B-action an A-B-module, we define for any two small dg-categories, $\mathcal{A}$ and $\mathcal{B}$, the category of dg $\mathcal{A}$-$\mathcal{B}$-bimodules to be $\mathrm{dgMod}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$. We note here that the symmetric monoidal closed structure on dgcat_k allows us to view bimodules as morphisms of dg-categories by the isomorphism

$$\begin{aligned} \mathrm{dgMod}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) &= \underline{\mathrm{dgcat}}_k(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{C}(k)) \\ &\cong \underline{\mathrm{dgcat}}_k(\mathcal{A}, \underline{\mathrm{dgcat}}_k(\mathcal{B}^{\mathrm{op}}, \mathcal{C}(k))) \\ &= \underline{\mathrm{dgcat}}_k(\mathcal{A}, \mathrm{dgMod}(\mathcal{B})). \end{aligned}$$

The image of a dg $\mathcal{A}$-$\mathcal{B}$-bimodule, E, is the dg-functor $\Phi_E(A) = E(A, -)$.

As a final note, we draw a connection between chain complexes and dg-modules over a ringoid that parallels the discussion of A-modules and the so-called Ab-modules above. Let A be a k-algebra and consider the category of chain complexes, $C(A)$. One can construct (see, e.g., Weibel (1994)) for any two chain complexes a chain complex of morphisms

$$\mathcal{C}(A)(C, D)^n = \prod_{m \in \mathbb{Z}} \operatorname{Mod} A \left(C^m, D^{m+n} \right)$$

with differential given by

$$d(f) = d_D \circ f + (-1)^{n+1} f \circ d_C.$$

Denoting by $\mathcal{C}(A)$ the category with objects chain complexes of A-modules and morphisms given by this complex, a similar translation shows that this is equivalent to the dg-category $\operatorname{dgMod}(\mathcal{A})$.

2.3 h-Projective DG-Modules

We say that a dg $\mathcal{A}$-module, N, is **acyclic** if $N(A)$ is an acyclic chain complex for all objects A of $\mathcal{A}$. A dg $\mathcal{A}$-module, M, is said to be **h-projective** if

$$H^0(\operatorname{dgMod}(\mathcal{A}))(M, N) := H^0(\operatorname{dgMod}(A)(M, N)) = 0$$

for every acyclic dg $\mathcal{A}$-module, N. The full dg-subcategory of $\operatorname{dgMod}(\mathcal{A})$ consisting of h-projectives will be called $\operatorname{h-proj}(\mathcal{A})$.

We always have a special class of h-projectives given by the representables, which we denote $h_A = \mathcal{A}(-, A)$, for if M is acyclic, then from the enriched Yoneda Lemma we have

$$H^0(\operatorname{dgMod}(A))(h_A, M) := H^0(\operatorname{dgMod}(\mathcal{A})(h_A, M)) \cong H^0(M(A)) = 0.$$

Noting that closure of $\operatorname{h-proj}(\mathcal{A})$ under homotopy equivalence follows immediately from the Yoneda Lemma applied to $H^0(\operatorname{dgMod}(A))$, we define $\overline{\mathcal{A}}$ to be the full dg-subcategory of h-proj$(\mathcal{A})$ consisting of the dg $\mathcal{A}$-modules homotopy equivalent to representables.

We will say an h-projective dg $\mathcal{A}$-$\mathcal{B}$-bimodule, E, is **right quasi-representable** if for every object A of $\mathcal{A}$ the dg $\mathcal{B}$-module $\Phi_E(A)$ is an object of $\overline{\mathcal{B}}$, and we will denote by h-proj$(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})^{\mathrm{rqr}}$ the full subcategory of h-proj$(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of all right quasi-representables.

2.4 The Derived Category of a DG-Category

By definition, a degree zero closed morphism

$$\eta \in Z^0(\mathrm{dgMod}(\mathcal{A}))(M, N)$$

satisfies

$$\eta(A) \in Z^0(\mathcal{C}(k)(M(A), N(A))) = C(k)(M(A), N(A))$$

for all objects A of $\mathcal{A}$. Hence we are justified in the following definitions:

(i) η is a **quasi-isomorphism** if $\eta(A)$ is a quasi-isomorphism of chain complexes for all objects A of $\mathcal{A}$, and

(ii) η is a **fibration** if $\eta(A)$ is a degree-wise surjective morphism of complexes for all objects A of $\mathcal{A}$.

Equipping $C(k)$ with the standard projective model structure (Hovey 1999, Section 2.3), these definitions endow $Z^0(\mathrm{dgMod}(\mathcal{A}))$ with the structure of a particularly nice cofibrantly generated model category (Toën 2007, Section 3). In analogy with the definition of the derived category of modules for a ring A, the **derived category of** $\mathcal{A}$ is defined to be the model category theoretic homotopy category,

$$\mathrm{D}(\mathcal{A}) = \mathrm{Ho}\left(Z^0(\mathrm{dgMod}(\mathcal{A}))\right) = Z^0(\mathrm{dgMod}(\mathcal{A}))[\mathcal{W}^{-1}]$$

obtained from localizing $Z^0(\mathrm{dgMod}(\mathcal{A}))$ at the class, $\mathcal{W}$, of quasi-isomorphisms.

It can be shown (Keller 1994, Section 3.5) that for every dg $\mathcal{A}$-module, M, there exists an h-projective, N, and a quasi-isomorphism $N \to M$, which one calls an **h-projective resolution of** M. Moreover, it is not difficult to see that any quasi-isomorphism between h-projective objects is in fact a homotopy equivalence. It follows that there is an equivalence of categories between $H^0(\text{h-proj}\,(\mathcal{A}))$ and $\mathrm{D}(\mathcal{A})$ for any small dg-category, $\mathcal{A}$.

It should be noted that this generalizes the notion of derived categories of modules over a k-algebra, A. Making the identification of $\mathcal{C}\,(A)$ and $\mathrm{dgMod}\,(\mathcal{A})$ as at the end of Section 2.2, where $\mathcal{A}$ is the ringoid associated to A, it is easy to recognize the categories $Z^0(\mathrm{dgMod}\,(\mathcal{A}))$, $H^0(\mathrm{dgMod}\,(\mathcal{A}))$, and $\mathrm{D}(\mathcal{A})$, as the categories $C\,(A)$, $K(A)$, the usual category up to homotopy, and the derived category of $\mathrm{Mod}\,A$, respectively. In the language of Lunts and Orlov (2010), $\text{h-proj}\,(\mathcal{A})$ is a dg-enhancement of $\mathrm{D}(\mathrm{Mod}\,A)$.

2.5 Tensor Products of DG-Modules

Let M be a dg $\mathcal{A}$-module, let N be a dg $\mathcal{A}^{\mathrm{op}}$-module, and let A, B be objects of $\mathcal{A}$. For ease of notation, we drop the functor notation $M(A)$ in favor of M_A and write $\mathcal{A}_{A,B}$ for the morphisms $\mathcal{A}(A,B)$. We have structure morphisms

$$M_{A,B} \in \mathcal{C}\,(k)\,(\mathcal{A}_{A,B}, \mathcal{C}\,(k)\,(M_B, M_A)) \cong \mathcal{C}\,(k)\,(M_B \otimes_k \mathcal{A}_{A,B}, M_A)$$

and

$$N_{A,B} \in \mathcal{C}\,(k)\,(\mathcal{A}_{A,B}, \mathcal{C}\,(k)\,(N_A, N_B)) \cong \mathcal{C}\,(k)\,(\mathcal{A}_{A,B} \otimes_k N_A, N_B)\,,$$

which give rise to a unique morphism

$$M_B \otimes_k A_{A,B} \otimes_k N_A \to M_A \otimes_k N_A \oplus M_B \otimes_k N_B$$

induced by the universal properties of the biproduct. The two collections of morphisms given by projecting onto each factor induce morphisms

$$\Xi_1, \Xi_2 \colon \bigoplus_{A,B \in \mathrm{Ob}(\mathcal{A})} M_B \otimes_k \mathcal{A}_{A,B} \otimes_k N_A \to \bigoplus_{\mathcal{C} \in \mathrm{Ob}(\mathcal{A})} M_C \otimes_k N_C,$$

and we define the tensor product of M and N to be the coequalizer in $C\left(k\right)$

$$\bigoplus_{(i,j)\in\mathbb{Z}^2} M_j \otimes_k \mathcal{A}_{A,B} \otimes_k N_A \underset{\Xi_2}{\overset{\Xi_1}{\rightrightarrows}} \bigoplus_{\ell\in\mathbb{Z}} M_\ell \otimes_k N_\ell \longrightarrow M \otimes_{\mathcal{A}} N \ .$$

It is routine to check that a morphism $M \to M'$ of right dg $\mathcal{A}$-modules induces by the universal property for coequalizers a unique morphism

$$M \otimes_{\mathcal{A}} N \to M' \otimes_{\mathcal{A}} N$$

yielding a functor

$$- \otimes_{\mathcal{A}} N \colon \ \mathrm{dgMod}\left(\mathcal{A}\right) \to \mathcal{C}\left(k\right).$$

One extends this construction to bimodules as follows. Given objects E of $\mathrm{dgMod}\left(\mathcal{A} \otimes \mathcal{B}\right)$ and F of $\mathrm{dgMod}\left(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{C}\right)$, we recall that we have associated to each a dg-functor

$$\Phi_E \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{dgMod}\left(\mathcal{B}\right) \ \text{ and } \Phi_F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{dgMod}\left(\mathcal{B}^{\mathrm{op}}\right)$$

by the symmetric monoidal closed structure on dgcat_k. For each pair of objects A of $\mathcal{A}$ and C of $\mathcal{C}$, we obtain dg-modules

$$\Phi_E(A) = E(A, -) \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\left(k\right) \ \text{ and } \Phi_F(C) = F(-, C) \colon \mathcal{B} \to \mathcal{C}\left(k\right)$$

and hence one may define the object $E \otimes_{\mathcal{B}} F$ of $\mathrm{dgMod}\left(\mathcal{A} \otimes \mathcal{C}\right)$ by

$$\left(E \otimes_{\mathcal{B}} F\right)(A, C) = \Phi_E(A) \otimes_{\mathcal{B}} \Phi_F(C).$$

One can show that by a similar argument to the original that a morphism $E \to E'$ of $\mathrm{dgMod}\left(\mathcal{A} \otimes \mathcal{B}\right)$ induces a morphism $E \otimes_{\mathcal{B}} F \to E' \otimes_{\mathcal{B}} F$ of $\mathrm{dgMod}\left(\mathcal{A} \otimes \mathcal{C}\right)$, and a morphism $F \to F'$ of $\mathrm{dgMod}\left(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{C}\right)$ induces a morphism $E \otimes_{\mathcal{B}} F \to E \otimes_{\mathcal{B}} F'$ of $\mathrm{dgMod}\left(\mathcal{A} \otimes \mathcal{C}\right)$.

Remark 2.5.1. Denote by $\mathcal{K}$ the dg-category with one object, $*$, and morphisms given by the chain complex

$$\mathcal{K}(*,*)^n = \begin{cases} k & n = 0 \\ 0 & n \neq 0 \end{cases}$$

with zero differential. This category serves as the unit of the symmetric monoidal structure on dgcat_k, so for small dg-categories, $\mathcal{A}$ and $\mathcal{C}$, we can always identify $\mathcal{A}$ with $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{C}$ with $\mathcal{K}^{\mathrm{op}} \otimes \mathcal{C}$. With this identification in hand, we obtain from taking $\mathcal{B} = \mathcal{K}$ in the latter construction a special case: Given a dg $\mathcal{A}^{\mathrm{op}}$-module, E, and a dg $\mathcal{C}$-module, F, we have a dg $\mathcal{A}$-$\mathcal{C}$-bimodule defined by the tensor product

$$(E \otimes F)(A, C) := (E \otimes_{\mathcal{K}} F)(A, C) = E(A) \otimes_k F(C).$$

2.6 Bimodules as Morphisms of Module Categories

Let E be a dg $\mathcal{A}$-$\mathcal{B}$-bimodule. Following Canonaco and Stellari (2015, Section 3), we can extend the associated functor Φ_E to a dg-functor

$$\widehat{\Phi_E}\colon \mathrm{dgMod}(\mathcal{A}) \to \mathrm{dgMod}(\mathcal{B})$$

defined by $\widehat{\Phi_E}(M) = M \otimes_{\mathcal{A}} E$. Similarly, we have a dg-functor in the opposite direction

$$\widetilde{\Phi_E}\colon \mathrm{dgMod}(\mathcal{B}) \to \mathrm{dgMod}(\mathcal{A})$$

defined by $\widetilde{\Phi_E}(N) = \mathrm{dgMod}(\mathcal{B})(\Phi_E(-), N)$.

For any dg-functor $G\colon \mathcal{A} \to \mathcal{B}$ we denote by Ind_G the extension of the dg-functor

$$\mathcal{A} \longrightarrow \mathcal{B} \xrightarrow{Y_{\mathcal{B}}} \mathrm{dgMod}(\mathcal{B})$$

and its right adjoint by Res_G. By way of the enriched Yoneda Lemma we see that for any object A of $\mathcal{A}$ and any dg $\mathcal{B}$-module, N,

$$\mathrm{Res}_G(N)(A) = \mathrm{dgMod}(\mathcal{B})(h_{GA}, N) \cong N(GA).$$

We record here some useful propositions regarding extensions of dg-functors.

Proposition 2.6.1 (Canonaco and Stellari (2015, Prop 3.2))**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be small dg-categories. Let $F\colon \mathcal{A} \to \mathrm{dgMod}\,(\mathcal{B})$ and $G\colon \mathcal{A} \to \mathcal{B}$ be dg-functors.*

(i) *$\widehat{F}$ is left adjoint to $\widetilde{F}$ (hence Ind_G is left adjoint to Res_G),*

(ii) *$\widehat{F} \circ Y_{\mathcal{A}}$ is dg-isomorphic to F and $H^0(\widehat{F})$ is continuous (hence $\mathrm{Ind}_G \circ Y_{\mathcal{A}}$ is dg-isomorphic to $Y_{\mathcal{B}} \circ G$ and $H^0(\mathrm{Ind}_G)$ is continuous),*

(iii) *$\widehat{F}(\text{h-proj}\,(\mathcal{A})) \subseteq \text{h-proj}\,(\mathcal{B})$ if and only if $F(A) \subseteq \text{h-proj}\,(B)$ (hence the essential image of $\text{h-proj}\,(\mathcal{A})$ under Ind_G lies in $\text{h-proj}\,(B)$),*

(iv) *$\mathrm{Res}_G(\text{h-proj}\,(\mathcal{B})) \subseteq \text{h-proj}\,(\mathcal{A})$ if and only if $\mathrm{Res}_G(\bar{B}) \subseteq \text{h-proj}\,(\mathcal{A})$; moreover, $H^0(\mathrm{Res}_G)$ is always continuous,*

(v) *$\mathrm{Ind}_G\colon\ \text{h-proj}\,(\mathcal{A}) \to \text{h-proj}\,(\mathcal{B})$ is a quasi-equivalence if G is a quasi-equivalence.*

Remark 2.6.2.

1. We note that for dg $\mathcal{A}$- and $\mathcal{A}^{\mathrm{op}}$-modules, M and N, part (i) implies that the dg-functors

$$- \otimes_{\mathcal{A}} N\colon\ \mathrm{dgMod}\,(\mathcal{A}) \to \mathcal{C}\,(k) \ \text{ and } M \otimes_{\mathcal{A}} -\colon\ \mathrm{dgMod}\,(\mathcal{A}^{\mathrm{op}}) \to \mathcal{C}\,(k)$$

have right adjoints

$$\widetilde{N}(C) = \mathcal{C}\,(k)\,(N(-), C) \text{ and } \widetilde{M}(C) = \mathcal{C}\,(k)\,(M(-), C),$$

respectively. As an immediate consequence of the enriched Yoneda Lemma

$$h_A \otimes_{\mathcal{A}} N \cong N(A) \text{ and } M \otimes_{\mathcal{A}} h^A \cong M(A)$$

holds for any object A of $\mathcal{A}$.

2. We denote by $\Delta_{\mathcal{A}}$ the dg $\mathcal{A}$-$\mathcal{A}$-bimodule corresponding to the Yoneda embedding, $Y_{\mathcal{A}}$, under the isomorphism

$$\mathrm{dgMod}\,(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}) \cong \underline{\mathrm{dgcat}}_k\,(\mathcal{A}, \mathrm{dgMod}\,(\mathcal{A}))\,.$$

It's clear that we have a dg-functor

$$\Delta_{\mathcal{A}} \otimes_{\mathcal{A}} -\colon \operatorname{dgMod}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}) \to \operatorname{dgMod}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$$

and for any dg $\mathcal{A}$-$\mathcal{A}$-bimodule, E, we see that

$$(\Delta_{\mathcal{A}} \otimes_{\mathcal{A}} E)(A, A') = h_A \otimes_{\mathcal{A}} E(-, A') \cong E(A, A')$$

implies that $\Delta_{\mathcal{A}} \otimes_{\mathcal{A}} E \cong E$.

When starting with an h-projective we have a very nice extension of dg-functors:

Proposition 2.6.3 (Canonaco and Stellari (2015, Lemma 3.4))**.** *For any h-projective dg $\mathcal{A}$-$\mathcal{B}$-bimodule, E, the extension of the associated functor*

$$\Phi_E\colon \mathcal{A} \to \operatorname{dgMod}(\mathcal{B})$$

factors through $\operatorname{h-proj}(\mathcal{B})$.

As a direct consequence of the penultimate proposition, one can view the extension of Φ_E as a dg-functor

$$\widehat{\Phi_E} = - \otimes_{\mathcal{A}} E\colon \operatorname{h-proj}(A) \to \operatorname{h-proj}(B).$$

That is to say, tensoring with an h-projective $\mathcal{A}$-$\mathcal{B}$-bimodule preserves h-projectives.

One essential result about dgcat_k comes from Töen's result on the existence, and description of, the internal Hom in its homotopy category.

Theorem 2.6.4 (Toën (2007, Thm 1.1), Canonaco and Stellari (2015, Section 4))**.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be objects of* dgcat_k*. There exists a natural bijection*

$$[\mathcal{A}, \mathcal{C}] \xleftrightarrow{1:1} \operatorname{Iso}\left(H^0(\operatorname{h-proj}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{C})^{\mathrm{rqr}})\right)$$

Moreover, the dg-category $\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{C}) := \operatorname{h-proj}(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{C})^{\mathrm{rqr}}$ *yields a natural bijection*

$$[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \xleftrightarrow{1:1} [\mathcal{A}, \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{C})]$$

proving that the symmetric monoidal category $\operatorname{Ho}(\mathrm{dgcat}_k)$ *is closed.*

Corollary 2.6.5 (Toën (2007, Thm 7.2),Canonaco and Stellari (2015, Cor. 4.2)). *Given two dg categories* $\mathcal{A}$ *and* $\mathcal{B}$*,* $\mathbf{R}\underline{\mathrm{Hom}}\left(\mathcal{A}, \text{h-proj}\left(\mathcal{B}\right)\right)$ *and* $\text{h-proj}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\right)$ *are isomorphic in* $\mathrm{Ho}\left(\mathrm{dgcat}_k\right)$*. Moreover, there exists a quasi-equivalence*

$$\mathbf{R}\underline{\mathrm{Hom}}_c\left(\text{h-proj}\left(\mathcal{A}\right), \text{h-proj}\left(\mathcal{B}\right)\right) \to \mathbf{R}\underline{\mathrm{Hom}}\left(\mathcal{A}, \text{h-proj}\left(B\right)\right).$$

To get a sense of the value of this result, let us recall one application from Toën (2007, Section 8.3). Let X and Y be quasi-compact and separated schemes over $\mathrm{Spec}\, k$. Recall the dg-model for $\mathrm{D}(\mathrm{Qcoh}\, X)$, $\mathcal{L}_{\mathrm{qcoh}}(X)$, is the $\mathcal{C}(k)$-enriched subcategory of fibrant and cofibrant objects in the injective model structure on $C(\mathrm{Qcoh}\, X)$.

Theorem 2.6.6 (Toën (2007, Thm. 8.3)). *Let* X *and* Y *be quasi-compact, quasi-separated schemes over* k*. Then there exists an isomorphism in* $\mathrm{Ho}\left(\mathrm{dgcat}_k\right)$

$$\mathbf{R}\underline{\mathrm{Hom}}_c\left(\mathcal{L}_{\mathrm{qcoh}}X, \mathcal{L}_{\mathrm{qcoh}}Y\right) \cong \mathcal{L}_{\mathrm{qcoh}}(X \times_k Y)$$

which takes a complex $E \in \mathcal{L}_{\mathrm{qcoh}}(X \times_k Y)$ *to the exact functor on the homotopy categories*

$$\begin{aligned} \Phi_E : \mathrm{D}(\mathrm{Qcoh}\, X) &\to \mathrm{D}(\mathrm{Qcoh}\, Y) \\ M &\mapsto \mathbf{R}\pi_{2*}\left(E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi_1^* M\right) \end{aligned}$$

Proof. The first part of the statement is exactly as in Toën (2007). The second part is implicit. □

2.7 Pretriangulated DG-Categories

In this section, we recall the definition of pretriangulated differential graded categories and provide a useful tool for proving that a dg-functor is a quasi-equivalence.

Definition 2.7.1. We say that a dg-category, $\mathcal{A}$, is pretriangulated if

(i) for all objects A of $\mathcal{A}$ and for all integers n there exists an object $A[n]$ representing the functor $h_A[n]$, and

(ii) for each morphism $f \in Z^0(\mathcal{A}(A_1, A_2))$ there exists an object cone(f) representing the pointwise cone functor

$$\operatorname{cone}(f_*)(A) = \operatorname{cone}\left(\mathcal{A}(A, A_1) \overset{f_*(A)}{\to} \mathcal{A}(A, A_2)\right)$$

In this case, the Yoneda embedding descends to a triangulated functor

$$H^0(Y_{\mathcal{A}})\colon H^0(\mathcal{A}) \to H^0(\mathrm{dgMod}\,(A)).$$

The following result will prove remarkably useful throughout.

Lemma 2.7.2 (Schwede and Shipley (2003, Lemma 2.2.1))**.** *Let $\mathcal{D}$ be a triangulated category with coproducts and let $\mathcal{K}$ be a set of compact objects. Then the following are equivalent:*

(i) the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{K}$ that is closed under coproducts is $\mathcal{D}$ itself,

(ii) An object D of $\mathcal{D}$ is trivial if and only if $\mathcal{D}(K, X[n]) = 0$ for all objects K of $\mathcal{K}$ and all integers n.

As a first application, we record a handy proposition. It is suspected that this is well known, but satisfactory references in the literature seem difficult to find.

Proposition 2.7.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be pretriangulated dg-categories. Assume that $H^0(\mathcal{A})$ and $H^0(\mathcal{B})$ each have a set of compact generators, $\{A_i\}_I$ and $\{B_j\}_J$. If $F\colon \mathcal{A} \to \mathcal{B}$ is a continuous dg-functor satisfying $F(\{A_i\}_I) = \{B_j\}_J$ and the structure morphism*

$$F_{A_{i_1}, A_{i_2}}\colon \mathcal{A}(A_{i_1}, A_{i_2}) \to \mathcal{B}(FA_{i_1}, FA_{i_2})$$

is a quasi-isomorphism for all $i_1, i_2 \in I$, then F is a quasi-equivalence.

Proof. We observe that it suffices to show that F is quasi-fully faithful. Indeed, if F is quasi-fully faithful, then the essential image of $H^0(\mathcal{A})$ under $H^0(F)$ is a triangulated subcategory of $H^0(\mathcal{B})$ that is closed under coproducts and contains the generators $\{B_j\}_J$ by assumption. Since $H^0(\mathcal{B})$ is the smallest such triangulated subcategory, it follows that the essential image of $H^0(\mathcal{A})$ is all of $H^0(\mathcal{B})$ and thus F is quasi-essentially surjective.

We break the argument into two pieces. The proof of each case is similar in style to the proof that F is quasi-essentially surjective above. In the first case, we show that the full dg-subcategory, $\mathcal{C}$, of $\mathcal{A}$ consisting of objects C such that

$$F_{A_i,C}\colon \mathcal{A}(A_i, C) \to \mathcal{B}(FA_i, FC)$$

is a quasi-isomorphism for all $i \in I$ satisfies $H^0(\mathcal{C}) = H^0(\mathcal{A})$, so that, being a full dg-subcategory of $\mathcal{A}$ with the same objects as $H^0(\mathcal{C})$, $\mathcal{C} = \mathcal{A}$. Having established this, we obtain a non-trivial full dg-subcategory, $\mathcal{D}$, of $\mathcal{A}$ consisting of objects D such that

$$F_{D,X}\colon \mathcal{A}(D, X) \to \mathcal{B}(FD, FX)$$

is a quasi-isomorphism for all objects X of $\mathcal{A}$. Once again we show that the subcategory $H^0(\mathcal{D}) = H^0(\mathcal{A})$. By the same argument, mutatis mutandis, this implies that $\mathcal{D} = \mathcal{A}$ and F is quasi-fully faithful.

Towards the first goal, we note that it suffices to show $H^0(\mathcal{C})$ is triangulated, closed under coproducts, and contains $\{A_i\}_I$. The latter condition is guaranteed by hypothesis. That $H^0(\mathcal{C})$ is closed under translation follows from the pretriangulated structure. Indeed, for any integer n, any $i \in I$, and any object C of $\mathcal{C}$ we have the isomorphisms

$$H^0(\mathcal{A}(A_i, C[n])) \cong H^0(\mathcal{A}(A_i, C)[n]) \cong H^n(\mathcal{A}(A_i, C))$$

and, similarly, $H^0(\mathcal{B}(FA_i, FC[n])) \cong H^n(\mathcal{B}(FA_i, FC))$. Now, for any distinguished triangle $C_1 \to C_2 \to X \to C_1[1]$ of $H^0(\mathcal{A})$ with C_1, C_2 objects of $\mathcal{C}$ we see that X is an object of $\mathcal{C}$ by applying the Five Lemma to the morphism of long exact sequences induced by the homological functors $h^0_{A_i}(-) := H^0(\mathcal{A})(A_i, -)$ and

$h^0_{FA_i}(-) := H^0(\mathcal{B})(FA_i, -)$

$$\begin{tikzcd}
\cdots \arrow[r] & h^0_{A_i}(C_1) \arrow[r] \arrow[d, "H^0(F_{A_i,C_1})"] & h^0_{A_i}(C_2) \arrow[r] \arrow[d, "H^0(F_{A_i,C_2})"] & h^0_{A_i}(X) \arrow[r] \arrow[d, "H^0(F_{A_i,X})"] & h^1_{A_i}(C_1) \arrow[r] \arrow[d, "H^1(F_{A_i,C_1})"] & h^1_{A_i}(C_2) \arrow[r] \arrow[d, "H^1(F_{A_i,C_2})"] & \cdots \\
\cdots \arrow[r] & h^0_{FA_i}(FC_1) \arrow[r] & h^0_{FA_i}(FC_2) \arrow[r] & h^0_{FA_i}(FX) \arrow[r] & h^1_{FA_i}(FC_1) \arrow[r] & h^1_{FA_i}(FC_2) \arrow[r] & \cdots
\end{tikzcd}$$

for each $i \in I$. Hence by equipping $H^0(\mathcal{C})$ with the distinguished triangles from $H^0(\mathcal{A})$ of the form $C_1 \to C_2 \to C_3 \to C_1[1]$ with the C_i objects of $\mathcal{C}$, $H^0(\mathcal{C})$ inherits the structure of a triangulated subcategory. Finally we note that because A_i and $FA_i \in \{B_j\}_J$ are compact, and the induced functor $H^0(F)$ commutes with direct sums, we have for any set, $\{C_\alpha\}$, of objects of $\mathcal{C}$ the isomorphism

$$H^0\left(\mathcal{A}\left(A_i, \bigoplus_\alpha C_\alpha\right)\right) \cong \bigoplus_\alpha H^0\left(\mathcal{A}\left(A_i, C_\alpha\right)\right) \cong \bigoplus_\alpha H^0\left(\mathcal{B}\left(FA_i, C_\alpha\right)\right)$$
$$\cong H^0\left(\mathcal{B}\left(FA_i, \bigoplus_\alpha FC_\alpha\right)\right) \cong H^0\left(\mathcal{B}\left(FA_i, F\left(\bigoplus_\alpha C_\alpha\right)\right)\right)$$

implies that $H^0(\mathcal{C})$ is closed under coproducts.

To see that $\mathcal{D} = \mathcal{A}$, we again observe that it suffices to show $H^0(\mathcal{D})$ is triangulated, closed under coproducts, and contains the generators, $\{A_i\}_I$. The latter condition follows from the fact that the category $\mathcal{C}$ contains $\{A_i\}_I$. For any object D of $\mathcal{D}$ and any object X of $\mathcal{A}$, the fact that translation is an auto-equivalence yields the natural isomorphisms

$$\mathcal{A}(D[n], X) \cong \mathcal{A}(D, X[-n]) \text{ and } \mathcal{B}(FD[n], FX) \cong \mathcal{B}(FD, FX[-n])$$

from which we obtain the isomorphism

$$H^0(\mathcal{A}(D[n], X)) \cong H^{-n}(\mathcal{A}(D, X)) \cong H^{-n}(\mathcal{B}(FD, FX)) \cong H^0(\mathcal{B}(FD[n], FX))$$

for all n. Hence $H^0(\mathcal{D})$ is closed under translations. Next we see that for any set of objects $\{D_\alpha\}$ of $\mathcal{D}$ and any object X of $\mathcal{A}$ we have the isomorphism

$$H^0\left(\mathcal{A}\left(\bigoplus_\alpha D_\alpha, X\right)\right) \cong \prod_\alpha H^0\left(\mathcal{A}\left(D_\alpha, X\right)\right) \cong \prod_\alpha H^0\left(\mathcal{B}\left(FD_\alpha, X\right)\right)$$
$$\cong H^0\left(\mathcal{B}\left(\bigoplus_\alpha FD_\alpha, X\right)\right) \cong H^0\left(\mathcal{B}\left(F\left(\bigoplus_\alpha D_\alpha\right), X\right)\right)$$

which implies that $H^0(\mathcal{D})$ is closed under coproducts. Finally, for any distinguished triangle $D_1 \to D_2 \to Z \to D_1[1]$ of $H^0(\mathcal{A})$ with D_1, D_2 objects of $\mathcal{D}$ we see that Z is an object of $\mathcal{D}$ by applying the Five Lemma to the morphism of long exact sequences induced by the cohomological functors $h_0^X(-) := H^0(\mathcal{A})(-, X)$ and $h_0^{FX}(-) := H^0(\mathcal{B})(-, FX)$

$$\begin{tikzcd}
\cdots \arrow[r] & h_0^X(D_2) \arrow[r] \arrow[d, "H^0(F_{D_2,X})"] & h_0^X(D_1) \arrow[r] \arrow[d, "H^0(F_{D_1,X})"] & h_1^X(Z) \arrow[r] \arrow[d, "H^1(F_{Z,X})"] & h_1^X(D_2) \arrow[r] \arrow[d, "H^1(F_{D_2,X})"] & h_1^X(D_1) \arrow[r] \arrow[d, "H^1(F_{D_1,X})"] & \cdots \\
\cdots \arrow[r] & h_0^{FX}(FD_X) \arrow[r] & h_0^{FX}(FD_1) \arrow[r] & h_1^{FX}(FZ) \arrow[r] & h_1^{FX}(FD_2) \arrow[r] & h_1^{FX}(FD_1) \arrow[r] & \cdots
\end{tikzcd}$$

for each $i \in I$. Hence by equipping $H^0(\mathcal{D})$ with the distinguished triangles

$$D_1 \to D_2 \to D_3 \to D_1[1]$$

of $H^0(\mathcal{A})$, where the D_i are objects of $\mathcal{D}$, inherits the structure of a triangulated subcategory. Therefore $\mathcal{D} = \mathcal{A}$ and F is quasi-fully faithful, as desired. $\square$

Chapter 3

Noncommutative Projective Schemes

Noncommutative projective schemes were introduced by Artin and Zhang (1994). In this section, we recall some of the basic definitions and results, as well as conditions that will appear in the sequel.

3.1 Graded Rings and Modules

Definition 3.1.1. Let G be a finitely-generated abelian group. We say that a k-algebra, A, is G-graded if there exists a decomposition as k-modules

$$A = \bigoplus_{g \in G} A_g$$

with $A_g A_h \subset A_{g+h}$. One says that A is **connected graded** if it is $\mathbb{Z}$-graded with $A_0 = k$ and $A_n = 0$ for $n < 0$.

Definition 3.1.2. We associate to a graded ring A the Grothendieck category of (left) G-graded modules, $\operatorname{Gr} A$, with morphisms $\operatorname{Gr} A(M, N)$ all degree preserving A-linear morphisms.

For a G-graded A-module, N, we write for $h \in G$

$$N(h) = \bigoplus_{g \in G} N_{g+h}$$

and we denote the graded module of morphism by

$$\underline{\mathrm{Gr}} A(M, N) := \bigoplus_{g \in G} \operatorname{Gr} A(M, N(g)).$$

Remark 3.1.3. In keeping with the notation above, we denote by A^{op} the opposite ring with multiplication reversed and we view the category of right G-graded A-modules as the category of left G-graded A^{op}-modules.

Definition 3.1.4. Let M be a graded A-module. We say that M has **right limited grading** if there exists some D such that $M_d = 0$ for all $D \leq d$. We define **left limited grading** analogously.

For a connected graded k-algebra, A, one has the bi-ideal

$$A_{\geq m} := \bigoplus_{n \geq m} A_n.$$

Definition 3.1.5. Let A be a finitely generated connected graded algebra. Recall that an element, m, of a module, M, is **torsion** if there is an n such that

$$A_{\geq n} m = 0.$$

We say that M is torsion if all its elements are torsion. We denote by $\operatorname{Tors} A$ the full subcategory of $\operatorname{Gr} A$ consisting of torsion modules.

3.2 Quotient Categories

Since the language for the objects in this section seems variable in the literature, we collect here some basic definitions and results from the theory of quotient categories so as to avoid any confusion. The standard reference is Gabriel (1962).

Definition 3.2.1. A full subcategory, $\mathcal{S}$, of an abelian category, $\mathcal{A}$ is called a **Serre subcategory** if for any short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

of $\mathcal{A}$, X is an object of $\mathcal{S}$ if and only if both X' and X'' are objects of $\mathcal{S}$.

Remark 3.2.2. It is easy to check that a Serre subcategory is an abelian category in its own right.

It is well known that for a left Noetherian, connected graded k-algebra, A, $\operatorname{Tors} A$ is a well-behaved Serre subcategory. In the commutative case, this of course covers the class of all finitely generated connected graded k-algebras. However, as noncommutative rings are generally less well behaved than their commutative counterparts, we note that even the noncommutative polynomial algebra $k\langle x, y\rangle$ is no longer left Noetherian (see, e.g. Goodearl and Warfield (2004, Exercise 1, p. 8)). The following proposition allows us to consider non-Noetherian rings.

Proposition 3.2.3. *Let A be a connected graded k-algebra. If A is finitely generated in positive degree, then* $\operatorname{Tors} A$ *is a Serre subcategory.*

Proof. Let $S = \{x_i\}_{i=1}^r$ be a set of generators for A as a k-algebra and let $d_i = \deg(x_i)$. Consider a short exact sequence

$$0 \to M' \to M \xrightarrow{p} M'' \to 0.$$

It's clear that if M is an object of $\operatorname{Tors} A$, then so are M' and M''. Hence it suffices to show that if M' and M'' are both objects of $\operatorname{Tors} A$, then so is M.

First assume that there exists some N such that for any $(X_1, X_2, \ldots, X_N) \in \prod_{i=1}^N S$ we have $(X_1 \cdots X_N)m = 0$. Let $d = \max(\{d_i\}_{i=1}^r)$ and take any $a \in A_{\geq dN}$. By assumption we can write $a = \sum_{i=1}^n \alpha_i a_i$ with $\alpha_i \in k$, each a_i of the form

$$a_i = X_{i,1}X_{i,2}\cdots X_{i,s_i},\ X_{i,j} \in S$$

and, for each i,

$$dN \leq \sum_{j=1}^{s_i} \deg(X_{i,j}) = \deg(a) \leq ds_i.$$

It follows that $N \leq s_i$ and hence $am = 0$. Thus it suffices to find such an N.

Fix an element $m \in M$. Since M'' is an object of $\operatorname{Tors} A$, there exists some n such that $A_{\geq n}p(m) = 0$ and hence $A_{\geq n}m \in M'$. In particular, if we let $T = \prod_{i=1}^n S$, then for any element $t = (X_1, X_2, \ldots, X_n) \in T$ we have an element $a_t = X_1X_2\cdots X_n \in A_{\geq n}$

and so $a_t m \in M'$. Let n_t be such that $A_{\geq n_t}(a_t m) = 0$ and take

$$N = 2\max(\{n_t\}_{t\in T} \cup \{n\}) + 1.$$

If we take any element $(X_1, X_2, \ldots, X_N) \in \prod_{i=1}^{N} S$, then we can form an element

$$a_t = X_{N-n}X_{N-n+1}\cdots X_N \in A_{\geq n}.$$

By construction, $a_t m \in M'$ and $a'_t = X_1X_2\cdots X_{N-n-1} \in A_{\geq n_t}$ since $n_t \leq N-n-1$. Therefore we have

$$0 = a'_t(a_t m) = (X_1X_2\cdots X_N)m,$$

as desired. □

Our only concern for Serre subcategories will be for the construction of a quotient. It can be shown that for any pair (X, Y) of objects of $\mathcal{A}$, equipping the collection of pairs of subobjects (X', Y') satisfying X/X', Y' both objects of $\mathcal{S}$ with the ordering

$$(X', Y') \leq (X'', Y'') \text{ if and only if } X'' \text{ is a subobject of } X', Y' \text{ is a subobject of } Y''$$

forms a directed system. One defines the **quotient of $\mathcal{A}$ by the Serre subcategory $\mathcal{S}$** to be the category $\mathcal{A}/\mathcal{S}$ with objects those of $\mathcal{A}$ and morphisms given by the colimit over this system

$$\mathcal{A}/\mathcal{S}(X, Y) = \operatorname{colim}_{(X',Y')} \mathcal{A}(X', Y/Y')$$

This quotient category comes equipped with a canonical projection functor

$$\pi \colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$$

which is the identity on objects and takes a morphism to its image in the colimit (Gabriel 1962, Cor. 1, III.1). The quotient is especially nice in the sense that the quotient is always abelian, π is always exact and, in the case that $\mathcal{A}$ is Grothendieck, the quotient is also Grothendieck.

In nice situations, this projection admits a section functor in the following sense.

Proposition 3.2.4. *Let $\mathcal{A}$ be an abelian category with injective envelopes and let $\mathcal{S}$ be a serre subcategory. The following are equivalent:*

(i) The functor π admits a fully faithful right adjoint, and

(ii) Every object M of $\mathcal{A}$ contains a subobject which is an object of $\mathcal{S}$ and is maximal amongst all such subobjects.

In this case, we say that $\mathcal{S}$ is a ***localizing subcategory****.*

Proof. This is Gabriel (1962, Cor. 1, III.3). □

Thanks to Proposition 3.2.3, $\operatorname{Tors} A$ is a coreflective Serre subcategory admitting a right adjoint, τ, to the inclusion, which takes a module M to its maximal torsion submodule, τM, whenever A is finitely generated in positive degree. As such, we can form the quotient.

Definition 3.2.5. For A a finitely generated graded k-algebra, denote the quotient of the category of graded A-modules by torsion as

$$\operatorname{QGr} A := \operatorname{Gr} A / \operatorname{Tors} A$$

Denote by $\omega : \operatorname{QGr} A \to \operatorname{Gr} A$ the right adjoint of π, and $Q := \omega\pi$.

Remark 3.2.6. In the sequel, it will be important to note that ω, being a fully faithful right adjoint to an exact functor, preserves injectives. In particular, this will guarantee that the adjunction lifts to a Quillen adjunction between $C(\operatorname{Gr} A)$ and $C(\operatorname{QGr} A)$, both equipped with the standard injective model structures. For details, see Hovey (2001).

The category $\operatorname{QGr} A$ is defined to be the quasi-coherent sheaves on the **noncommutative projective scheme** X.

Remark 3.2.7. Note that, traditionally speaking, X is not a space, in general. In the case A is commutative and finitely-generated by elements of degree 1, then a famous result of Serre says that X is $\operatorname{Proj} A$.

Proposition 3.2.8. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{S}$ be a Serre subcategory. For any object X of $\mathcal{A}$, the following are equivalent:*

1. *Given an exact sequence*

$$0 \longrightarrow K \xrightarrow{\ker f} Z \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with K and C objects of $\mathcal{S}$, the canonical morphism

$$h_X(f)\colon \mathcal{A}(Y,X) \to \mathcal{A}(Z,X)$$

is an isomorphism,

2. *The maximal $\mathcal{S}$-subobject of X is the zero object and any short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with C an object of $\mathcal{S}$ splits, and

3. *For any object Y of $\mathcal{A}$, $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ induces an isomorphism*

$$\mathcal{A}(Y,X) \cong \mathcal{A}/\mathcal{S}(\pi(Y),\pi(X)).$$

We say that an object X of $\mathcal{A}$ is $\mathcal{S}$-closed if any of these conditions are satisfied.

Proof. First assume (1). Denote by $\imath\colon X_{\mathcal{S}} \to X$ the maximal $\mathcal{S}$-subobject of X. If we let $p = \operatorname{coker}\imath\colon X \to X/X_{\mathcal{S}}$, then we have the exact sequence

$$0 \longrightarrow X_{\mathcal{S}} \xrightarrow{\imath=\ker p} X \xrightarrow{p} X/X_{\mathcal{S}} \xrightarrow{\operatorname{coker} p} 0 \longrightarrow 0.$$

By assumption the morphism

$$h_X(p)\colon \mathcal{A}(X/X_{\mathcal{S}}) \to \mathcal{A}(X,X)$$

is an isomorphism because both the zero object and $X_{\mathcal{S}}$ are objects of $\mathcal{S}$, hence p admits a section $s\colon X/X_{\mathcal{S}} \to X$. It follows from

$$0 = p \circ \imath = s \circ p \circ \imath = \imath$$

that $X_{\mathcal{S}}$ is the zero object. Similarly, if we have any short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{p} C \to 0$$

with C an object of $\mathcal{S}$, then we obtain by assumption an isomorphism

$$h_X(f)\colon \mathcal{A}(Y,X) \to \mathcal{A}(X,X)$$

which provides a section $s\colon Y \to X$ of f splitting the sequence. This establishes (2).

Assume (2). Let Y be an object of $\mathcal{A}$. We first show that the structure morphism

$$\pi_{Y,X}\colon \mathcal{A}(Y,X) \to \mathcal{A}/\mathcal{S}(\pi Y, \pi X)$$

is surjective. Given a morphism $f \in \mathcal{A}/\mathcal{S}(\pi Y, \pi X)$, we may lift by the definition to some morphism $f'\colon Y' \to X/X'$ where Y/Y' and X' are objects of $\mathcal{S}$. We note that, by assumption, $X' \subseteq X_{\mathcal{S}} = 0$, so $X/X' = X$ and we obtain the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r, "\imath"] \arrow[d, "f'"] & Y \arrow[r] \arrow[d, "f''"] & Y/Y' \arrow[r] \arrow[d, dashed, "\exists! h"] & 0 \\
0 \arrow[r] & X \arrow[r, "\imath'"] & Y \coprod_{Y'} X \arrow[r] & \left(Y \coprod_{Y'} X\right)/X \arrow[r] & 0
\end{tikzcd}$$

with the induced map of cokernels an isomorphism. The bottom row splits by assumption, giving a retract $r\colon Y \coprod_Y' X \to X$ of $\imath'$, and hence a morphism $r \circ f'' \in \mathcal{A}(Y,X)$. By the colimit definition of the morphisms, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{A}(Y,X) \arrow[r, "h_X(\imath)"] \arrow[dr, "\pi_{Y,X}"'] & \mathcal{A}(Y',X) \arrow[d, "\pi_{Y',X}"] \\
& \mathcal{A}/\mathcal{S}(\pi Y, \pi X)
\end{tikzcd}$$

with

$$\pi(h_X(\imath)(r \circ f'')) = \pi(r \circ f'' \circ \imath) = \pi(r \circ \imath' \circ f') = \pi(f') = f$$

from which it follows that $\pi_{Y,X} \colon \mathcal{A}(Y, X) \to \mathcal{A}/\mathcal{S}(\pi Y, \pi X)$ is surjective. To see that $\pi_{Y,X}$ is injective, we observe that a morphism $f \colon Y \to X$ satisfies $\pi(f) = 0$ if and only if in the factorization

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "\operatorname{coim} f"'] & & X \\
& f(Y) \arrow[ur, "\operatorname{im} f"'] &
\end{tikzcd}$$

the object $f(Y)$ is an object of $\mathcal{S}$. However, by maximality, the monomorphism $\operatorname{im} f$ factors through the monic $X_{\mathcal{S}} \xrightarrow{0} X$, and thus

$$f = \mathrm{imf} \circ \operatorname{coim} f = 0.$$

This establishes (3).

Finally, assume (3). Given an exact sequence

$$0 \longrightarrow K \xrightarrow{\ker f} Z \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with K and C objects of $\mathcal{S}$, we see that $\pi(f) \in \mathcal{A}/\mathcal{S}(\pi Z, \pi Y)$ is an isomorphism, hence

$$h_{\pi X}(\pi f) \colon \mathcal{A}/\mathcal{S}(\pi Y, \pi X) \to \mathcal{A}/\mathcal{S}(\pi Z, \pi X)$$

is an isomorphism. Because π is a functor we obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{A}(Y, X) \arrow[r, "\pi_{Y,X}"] \arrow[d, "h_X(f)"] & \mathcal{A}/\mathcal{S}(\pi Y, \pi X) \arrow[d, "h_{\pi X}(\pi f)"] \\
\mathcal{A}(Z, X) \arrow[r, "\pi_{Z,X}"] & \mathcal{A}/\mathcal{S}(\pi Z, \pi X)
\end{tikzcd}$$

Since $\pi_{Y,X}$ and $\pi_{Z,X}$ are isomorphisms by assumption, it follows that $h_X(f)$ is also an isomorphism. This establishes (1). □

As an immediate consequence of the Yoneda Lemma and condition 3 of Proposition 3.2.8, loosely speaking, $\operatorname{QGr} A$ is just the full subcategory of $\operatorname{Tors} A$-closed objects.

Corollary 3.2.9. *An object M of* $\operatorname{Gr} A$ *is* $\operatorname{Tors} A$*-closed if and only if $M \cong QM$. Consequently, π preserves* $\operatorname{Tors} A$*-closed injectives.*

Proof. The second statement is immediate from the isomorphism of adjunction, $\operatorname{Gr} A(-, I) \cong \operatorname{QGr} A(\pi(-), \pi I)$. □

In the special case that a localizing subcategory is closed under injective envelopes, we have the following characterization of injectives within the ambient abelian category.

Proposition 3.2.10. *Let $\mathcal{A}$ be an abelian category with injective envelopes, and let $\mathcal{S}$ be a localizing subcategory. For each object X of $\mathcal{A}$ denote by $X_{\mathcal{S}}$ the maximal $\mathcal{S}$-subobject. If $\mathcal{S}$ is closed under injective envelopes, then for every injective I of $\mathcal{A}$*

$$I \cong I_{\mathcal{S}} \oplus \omega\pi I.$$

Proof. Let $I_{\mathcal{S}} \to E$ be an injective envelope. Since I is injective we have an extension over the inclusion of the maximal $\mathcal{S}$-subobject

$$\begin{tikzcd} 0 \arrow[r] & I_{\mathcal{S}} \arrow[r] \arrow[d] & E \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

and this extension is necessarily monic because injective envelopes are essential monomorphisms. By maximality of $I_{\mathcal{S}}$ amongst all $\mathcal{S}$-subobjects of I, it follows that $I_{\mathcal{S}} = E$ is injective. Denoting by ε the unit of the adjunction $\pi \dashv \omega \colon \mathcal{A} \rightleftarrows \mathcal{A}/\mathcal{S}$ the exact sequence

$$0 \longrightarrow I_{\mathcal{S}} \longrightarrow I \xrightarrow{\varepsilon(I)} \omega\pi A I \longrightarrow 0$$

splits, as desired. □

We record here as a corollary a more explicit version of Artin and Zhang (1994, Prop 7.1 (5)), which states that every injective object of $\operatorname{Gr} A$ is of the form $I_1 \oplus I_2$, with I_1 a torsion-free injective and I_2 an injective torsion module. This will be useful for computations involving total derived functors in the sequel.

Corollary 3.2.11. *Let A be a left Noetherian, connected graded k-algebra. Every injective I of* $\operatorname{Gr} A$ *is isomorphic to* $\tau_A I \oplus Q_A I$.

Proof. By Artin and Zhang (1994, Prop 2.2) any essential extension of a torsion module is torsion. Now apply Proposition 3.2.10. □

3.3 Sheaf Cohomology

The funtor Q admits a more geometrically pleasing interpretation, which will serve to help interpret the somewhat onerous conditions in the sequel. We will often refer to the image of A in $\operatorname{QGr} A$ as $\mathcal{O}_X$, thinking of this as the structure sheaf on the noncommutative projective scheme X. Following Artin and Zhang (1994), one defines sheaf cohomology of a quasi-coherent sheaf $\mathcal{M} = \pi M$ to be

$$\underline{H}^i(\mathcal{M}) := \underline{\operatorname{Ext}}^i_{\operatorname{QGr} A}(\mathcal{O}_X, \mathcal{M})$$

and the un-graded sheaf cohomology by

$$H^i(\mathcal{M}) := \underline{H}^i(\mathcal{M})_0.$$

For the Ext-computations, generally one takes an injective resolution I of $\omega\mathcal{M}$ in $\operatorname{Gr} A$ then computes

$$\underline{H}^i(\mathcal{M}) = H^i\underline{\mathrm{QGr}}A(\mathcal{O}_X, \pi I) \cong H^i\underline{\mathrm{Gr}}A(A, QI) \cong H^i(QI) \cong \mathbf{R}^i Q(M).$$

In some sense, the functor Q should therefore be like the usual global sections functor.

On the other hand, one can also give more explicit descriptions of Q and τ.

Proposition 3.3.1. *Let A be a finitely generated connected graded k-algebra and let M be a graded A-module. Then*

$$\tau M = \operatorname{colim}_n \underline{\mathrm{Gr}} A(A/A_{\geq n}, M)$$

$$QM = \operatorname{colim}_n \underline{\mathrm{Gr}} A(A_{\geq n}, M).$$

Proof. This is standard localization theory, see Stenström (1975). □

3.4 Noncommutative Biprojective Schemes

In studying questions of kernels and bimodules, we will have to move outside the realm of $\mathbb{Z}$-gradings. While one can generally treat G-graded k-algebras in our analysis, we limit the scope a bit and only consider $\mathbb{Z}^2$-gradings of the following form.

Definition 3.4.1. Let A and B be connected graded k-algebras. The tensor product $A \otimes_k B$ will be equipped with its natural bi-grading

$$(A \otimes_k B)_{n_1,n_2} = A_{n_1} \otimes_k B_{n_2}.$$

A **bi-bi module** for the pair (A, B) is a $\mathbb{Z}^2$-graded $A \otimes_k B$ module.

Remark 3.4.2. As noted in the remarks above Van den Bergh (2001, Lemma 4.1), the notion of A-torsion and B-torsion bi-bi modules is well-defined provided that A and B are finitely generated as k-algebras. From this point on, unless stated otherwise, all of our k-algebras will be assumed to be finitely generated.

There are a few notions of torsion for a bi-bi module that one could use, but we take the following.

Definition 3.4.3. Let A and B be finitely generated, connected graded k-algebras, and let M be a bi-bi A-B module. We say that M is **torsion** if it lies in the smallest Serre subcategory containing A-torsion bi-bi modules and B-torsion bi-bi modules.

Lemma 3.4.4. *Let A and B be finitely generated, connected graded k-algebras. A bi-bi module M is torsion if and only if there exists n_1, n_2 such that*

$$(A \otimes B)_{\geq n_1, \geq n_2} m = 0$$

for all $m \in M$.

Proof. For necessity, note that if M is A-torsion, then $(A \otimes B)_{\geq n, \geq 0} m = 0$ for some n for each $m \in M$. Similarly if M is B-torsion then $(A \otimes B)_{\geq 0, \geq n} M = 0$ for some n. So it suffices to show that if

$$(A \otimes B)_{\geq n_1, \geq n_2} m = 0, \forall m \in M$$

then it lies in the Serre category generated by A and B torsion. Let $\tau_B M$ be the B-torsion submodule of M and consider the quotient $M/\tau_B M$. For $m \in M$, we have $A_{\geq n_1} m$ is B-torsion, so its image in the quotient $M/\tau_B M$ is A-torsion. Consequently, $M/\tau_B M$ is A-torsion itself and M is an extension of B-torsion and A-torsion. □

One can form the quotient category

$$\operatorname{QGr} A \otimes_k B := \operatorname{Gr} A \otimes_k B / \operatorname{Tors} A \otimes_k B.$$

Lemma 3.4.5. *The quotient functor*

$$\pi : \operatorname{Gr} A \otimes_k B \to \operatorname{QGr} A \otimes_k B$$

has a fully faithful right adjoint

$$\omega : \operatorname{QGr} A \otimes_k B \to \operatorname{Gr} A \otimes_k B$$

with

$$QM := \omega \pi M = \operatorname{colim}_{n_1, n_2} \underline{\operatorname{Gr}}(A \otimes_k B)(A_{\geq n_1} \otimes_k B_{\geq n_2}, M)$$

Proof. This is just an application of Gabriel (1962, Cor. 1, III.3). □

Corollary 3.4.6. *We have an isomorphism*

$$Q_{A\otimes_k B} \cong Q_A \circ Q_B \cong Q_B \circ Q_A$$

Proof. This follows from Lemma 3.4.5 using tensor-Hom adjunction. □

We also have the following standard triangles of derived functors.

Lemma 3.4.7. *Let A and B be finitely generated connected graded algebras. Then, we have natural transformations*

$$\mathbf{R}\tau \to \mathrm{Id} \to \mathbf{R}Q$$

which when applied to any graded module M gives an exact triangle

$$\mathbf{R}\tau M \to M \to \mathbf{R}QM.$$

Proof. Before we begin the proof, we clarify the statement. The conclusions hold for graded A (or B) modules and for bi-bi modules. Due to the formal properties, it is economical to keep the wording of the theorem as so since any reasonable interpretation yields a true statement.

For the case of graded A modules, this is well-known, see Bondal and Van den Bergh (2003, Property 4.6). For the case of bi-bi $A \otimes_k B$ modules, the natural transformations are obvious. For each M, the sequence

$$0 \to \tau M \to M \to QM$$

is exact. It suffices to prove that if $M = I$ is injective, then the whole sequence is actually exact. Here one can use the system of exact sequences

$$0 \to A_{\geq n_1} \otimes_k B_{\geq n_2} \to A \otimes_k B \to (A \otimes_k B)/A_{\geq n_1} \otimes_k B_{\geq n_2} \to 0$$

and exactness of $\mathrm{Hom}(-, I)$ plus Lemma 3.4.4 to get exactness. □

3.5 Cohomological Assumptions

In general, good behavior of $\mathrm{QGr}\, A$ occurs with some homological assumptions on the ring A. We recall two such common assumptions.

Definition 3.5.1. Let A be a connected graded k-algebra. Following Van den Bergh (2001), we say that A is **Ext-finite** if for each $n \geq 0$ the ungraded Ext-groups are finite dimensional

$$\dim_k \mathrm{Ext}^n_A(k, k) < \infty.$$

Remark 3.5.2. The Ext's are taken in the category of left A-modules, a priori. Moreover, as noted in the opening remarks of Bondal and Van den Bergh (2003, Section 4.1), if A is Ext-finite, then A is finitely presented.

Definition 3.5.3. Following Artin and Zhang (1994), given a graded left module M, we say A satisfies $\chi^\circ(M)$ if $\underline{\mathrm{Ext}}^n_A(k, M)$ has right limited grading for each $n \geq 0$.

Remark 3.5.4. The equivalence of these two definitions is Artin and Zhang (1994, Proposition 3.8 (1)).

We recall some basic results on Ext-finiteness, essentially from Van den Bergh (2001, Section 4).

Proposition 3.5.5. *Assume that A and B are Ext-finite. Then*

1. *the ring $A \otimes_k B$ is Ext-finite.*
2. *the ring A^{op} is Ext-finite.*

Proof. See Van den Bergh (2001, Lemma 4.2) and the discussion preceeding it. $\square$

Proposition 3.5.6. *Assume that A is Ext-finite. Then $\mathbf{R}\tau_A$ and $\mathbf{R}Q_A$ both commute with coproducts.*

Proof. See Van den Bergh (2001, Lemma 4.3) for $\mathbf{R}\tau_A$. Since coproducts are exact, using the triangle

$$\mathbf{R}\tau_A M \to M \to \mathbf{R}Q_A M$$

we see that $\mathbf{R}\tau_A$ commutes with coproducts if and only if $\mathbf{R}Q_A$ commutes with coproducts. $\square$

Corollary 3.5.7. *Let A and B be finitely generated, connected graded k-algebras, and let P be a chain complex of bi-bi $A \otimes_k B$ modules. Assume $\mathbf{R}Q_A$ commutes with coproducts. Then, $\mathbf{R}Q_A P$ is naturally also a chain complex of bi-bi modules. In particular, if A is Ext-finite, $\mathbf{R}Q_A P$ has a natural bi-bi structure.*

Proof. Note we already have an A-module structure so we only need to provide a $\mathbb{Z}^2$ grading and a B-action. If we write

$$P = \bigoplus_{v \in \mathbb{Z}} P_{*,v}$$

as a direct sum of left graded A-modules, then we set

$$(\mathbf{R}Q_A P)_{u,v} := (\mathbf{R}Q_A(P_{*,v}))_u.$$

The B module structure is precomposition with the B-action on P. The only non-obvious condition of the bi-bi structure is that

$$\mathbf{R}Q_A P = \bigoplus_{u,v} (\mathbf{R}Q_A P)_{u,v}$$

which is equivalent to pulling the coproduct outside of $\mathbf{R}Q_A$. We can do this for Ext-finite A thanks to Proposition 3.5.6. $\square$

Corollary 3.5.8. *Assume that A and B are left Noetherian, and that $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_B$ both commute with coproducts. There exist natural morphisms of bimodules*

$$\beta^l_P : \mathbf{R}Q_A P \to \mathbf{R}Q_{A \otimes_k B} P$$

$$\beta^r_P : \mathbf{R}Q_B P \to \mathbf{R}Q_{A \otimes_k B} P.$$

Proof. Thanks to Corollary 3.5.7, we see that the question is well-posed. We handle the case of β_P^l and note that case of β_P^r is the same argument, mutatis mutandis.

First we make some observations about objects of $\mathrm{Gr}\,(A \otimes_k B)$. If we regard such an object, E, as an A-module, the A-action is

$$a \cdot e = (a \otimes 1) \cdot e$$

and we can view $\tau_A E$ as the elements e of E for which

$$a \cdot e = (a \otimes 1) \cdot e = 0$$

whenever $a \in A_{\geq m}$ for some $m \in \mathbb{Z}$. As such, $\tau_A E$ inherits a bimodule structure from E and $\mathbb{Z}^2$-grading $(\tau_A E)_{u,v} = (\tau_A E_{*,v})_u$ coming from the decomposition

$$\tau_A E = \tau_A \bigoplus_v E_{*,v} \cong \bigoplus_v \tau_A E_{*,v}.$$

Thanks to Lemma 3.4.4, we can view $\tau_{A \otimes_k B} E$ as the elements e of E for which there exists integers m and n such that $a \otimes b \cdot e = 0$ for all $a \in A_{\geq m}$ and $b \in B_{\geq n}$. From this viewpoint it's clear that

$$a \otimes b \cdot e = (1 \otimes b) \cdot (a \otimes 1 \cdot e)$$

implies $\tau_A E$ includes into $\tau_{A \otimes_k B} E$.

We equip $C\,(\mathrm{Gr}\,A)$ with the injective model structure and use the methods of model categories to compute the derived functors (see Hovey (2001) for more details). Since we can always replace P by a quasi-isomorphic fibrant object, we can assume that P^n is an injective graded $A \otimes_k B$-module. Moreover, the fact that the canonical morphisms $A \to A \otimes_k B$ is flat implies that the associated adjunction is Quillen, and hence P is fibrant when regarded as an object of $C\,(\mathrm{Gr}\,A)$. Since Q_A preserves injectives, it follows that each $Q_A P^n$ is an injective object of $\mathrm{Gr}\,A$. It's clear from the fact that $\tau_A P^n$ is an $A \otimes_k B$-module that

$$0 \to \tau_A P^n \to P^n \to P^n / \tau_A P^n \to 0$$

is an exact sequence of $\mathrm{Gr}(A \otimes_k B)$ for each n. Moreover, by Lemma 3.2.11 we have $P^n/\tau_A P^n \cong Q_A P^n$. We thus define $(\beta_P^l)^n$ to be the epimorphism induced by the universal property for cokerenels as in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tau_A P^n \arrow[r] \arrow[d] & P^n \arrow[r, "\varepsilon_A(P^n)"] \arrow[d, "\mathrm{id}_{P^n}"] & Q_A P^n \arrow[r] \arrow[d, dashed, "\exists!(\beta_P^l)^n"] & 0 \\
0 \arrow[r] & \tau_{A\otimes_k B} P^n \arrow[r] & P^n \arrow[r, "\varepsilon_{A\otimes_k B}(P^n)"] & Q_{A\otimes_k B} P^n \arrow[r] & 0
\end{tikzcd}$$

To see that β actually defines a morphism of complexes, we have by naturality of ε_A, $\varepsilon_{A\otimes_k B}$, and the commutative diagram defining $(\beta_P^l)^n$ above

$$\begin{aligned}
(\beta_P^l)^{n+1} \circ Q_A(d_P^n) \circ \varepsilon_A(P^n) &= (\beta_P^l)^{n+1} \circ \varepsilon_A(P^{n+1}) \circ d_P^n \\
&= \varepsilon_{A\otimes_k B}(P^{n+1}) \circ d_P^n \\
&= Q_{A\otimes_k B}(d_P^n) \circ \varepsilon_{A\otimes_k B}(P^n) \\
&= Q_{A\otimes_k B}(d_P^n) \circ (\beta_P^l)^n \circ \varepsilon_A(P^n)
\end{aligned}$$

implies

$$(\beta_P^l)^{n+1} \circ Q_A(d_P^n) = Q_{A\otimes_k B}(d_P^n) \circ (\beta_P^l)^n$$

because $\varepsilon_{A\otimes_k B}(P^n)$ is epic. Hence we have a morphism

$$\beta_P^l \colon \mathbf{R}Q_A P = Q_A P \to Q_{A\otimes_k B} P = \mathbf{R}Q_{A\otimes_k B} P.$$

For naturality, we note that, as the fibrant replacement is functorial, if we have a morphism of bi-bi modules, then there is an induced morphism $\varphi\colon P_1 \to P_2$ of complexes between the replacements and for each n a commutative diagram

$$\begin{tikzcd}
P_1^n \arrow[r, "\varepsilon_A(P_1^n)"] \arrow[d, "\varphi^n"] & Q_A P_1^n \arrow[r, "(\beta_{P_1}^l)^n"] \arrow[d, "Q_A(\varphi^n)"] & Q_{A\otimes_k B} P_1^n \arrow[d, "Q_{A\otimes_k B}(\varphi^n)"] \\
P_2^n \arrow[r, "\varepsilon_A(P_2^n)"] & Q_A P_2^n \arrow[r, "(\beta_{P_2}^l)^n"] & Q_{A\otimes_k B} P_2^n
\end{tikzcd}$$

The left square commutes by naturality of ε_A and the right square commutes because

$$\begin{aligned}(\beta^l_{P_2})^n \circ Q_A(\varphi^n) \circ \varepsilon_A(P^n_1) &= (\beta^l_{P_2})^n \circ \varepsilon_A(P^n_2) \circ \varphi^n \\ &= \varepsilon_{A\otimes_k B}(P^n_2) \circ \varphi^n \\ &= Q_{A\otimes_k B}(\varphi^n) \circ \varepsilon_{A\otimes_k B}(P^n_1) \\ &= Q_{A\otimes_k B}(\varphi^n) \circ (\beta^l_{P_1})^n \circ \varepsilon_A(P^n_1)\end{aligned}$$

and $\varepsilon_A(P^n_1)$ is epic. $\square$

Proposition 3.5.9. *Assume that A and B are left Noetherian and Ext-finite. Then, we have natural quasi-isomorphisms*

$$\mathbf{R}Q_B(\beta^l_P) : \mathbf{R}Q_B(\mathbf{R}Q_A P) \to \mathbf{R}Q_{A\otimes_k B}P$$

$$\mathbf{R}Q_A(\beta^r_P) : \mathbf{R}Q_A(\mathbf{R}Q_B P) \to \mathbf{R}Q_{A\otimes_k B}P.$$

Consequently, β^l_P (respectively β^r_P) is an isomorphism if and only if $\mathbf{R}Q_A P$ (respectively $\mathbf{R}Q_B P$) is Q_B (respectively Q_A) torsion-free.

Proof. As above, we can replace P with a quasi-isomorphic fibrant object, so it suffices to assume that P is fibrant. We see from Corollary 3.4.6 that

$$\mathbf{R}Q_{A\otimes_k B}P \cong Q_{A\otimes_k B}P \cong Q_B \circ Q_A P \cong \mathbf{R}(Q_B \circ Q_A)P$$

The result now follows from the natural isomorphism (see, e.g., Hovey (1999, Theorem 1.3.7))

$$\mathbf{R}Q_B \circ \mathbf{R}Q_A \to \mathbf{R}(Q_B \circ Q_A)$$

$\square$

In the case that $A = B$, there is a particular bi-bi module of interest.

Definition 3.5.10. Let Δ_A be the A-A bi-bi module with

$$(\Delta_A)_{i,j} = A_{i+j}$$

and the natural left and right A actions. If the context is clear, we will often simply write Δ.

Using the standard homological assumptions above, one has better statements for $P = \Delta$.

Proposition 3.5.11. *Let A be left (respectively, right) Noetherian and assume that the condition $\chi^\circ(A)$ holds (respectively, as an A^{op}-module). Then the morphism β^l_Δ (respectively, β^r_Δ) of Corollary 3.5.8 is a quasi-isomorphism.*

Proof. We have a triangle in $\mathrm{D}(\mathrm{Gr}\, A \otimes_k A^{\mathrm{op}})$

$$\mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta) \to \mathbf{R}Q_A\Delta \to \mathbf{R}Q_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta) \to \mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta)[1].$$

By Proposition 3.5.9, $\mathbf{R}Q_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta) \cong \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta$, so it suffices to show that we have $\mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta) = 0$. Applying $\mathbf{R}\tau_{A^{\mathrm{op}}}$ to the triangle

$$\mathbf{R}\tau_A\Delta \to \Delta \to \mathbf{R}Q_A\Delta \to \mathbf{R}\tau_A\Delta[1]$$

we obtain the triangle

$$\mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}\tau_A\Delta) \to \mathbf{R}\tau_{A^{\mathrm{op}}}\Delta \to \mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta) \to \mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}\tau_A\Delta)[1]$$

and so we are reduced to showing that

$$\mathbf{R}\tau_A\Delta \cong \mathbf{R}\tau_{A\otimes_k A^{\mathrm{op}}}\Delta \cong \mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}\tau_A\Delta)$$

which then implies that $\mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}Q_A\Delta) = 0$, as desired.

First we note that for any bi-bi module, P, the natural morphism

$$\mathbf{R}\tau_{A^{\mathrm{op}}}P \to P$$

is a quasi-isomorphism if and only if the natural morphism

$$\mathbf{R}\tau_{A^{\mathrm{op}}}P_{x,*} \to P_{x,*}$$

is a quasi-isomorphism. Moreover, for a right A-module, M, if $H^j(M)$ is right limited for each j then $\mathbf{R}\tau_{A^{\mathrm{op}}} M \to M$ is a quasi-isomorphism, so it suffices to show that $(\mathbf{R}^j \tau_A \Delta)_{x,*}$ has right limited grading for each x and j. Now, by Artin and Zhang (1994, Cor. 3.6 (3)), for each j

$$\mathbf{R}^j \tau_A(\Delta)_{x,y} = \mathbf{R}^j \tau_A(\Delta_{*,y})_x = \mathbf{R}^j \tau_A(A(y))_x = 0$$

for fixed x and sufficiently large y. This implies that the natural morphism

$$\mathbf{R}\tau_{A^{\mathrm{op}}}(\mathbf{R}\tau_A(\Delta)_{x,*}) \to \mathbf{R}\tau_A \Delta_{x,*}$$

is a quasi-isomorphism, as desired. □

Similar hypotheses of Proposition 3.5.11 will appear often, so we attach a name.

Definition 3.5.12. Let A and B be connected graded k-algebras. If A is Ext-finite, left and right Noetherian, and satisfies $\chi^\circ(A)$ and $\chi^\circ(A^{\mathrm{op}})$ then we say that A is **delightful**. If A and B are both delightful, then we say that A and B form a **delightful couple**.

3.6 Segre Products

Definition 3.6.1. Let A and B be connected graded k-algebras. The **Segre product** of A and B is the graded k-algebra

$$A \times_k B = \bigoplus_{0 \le i} A_i \otimes_k B_i.$$

Proposition 3.6.2. *If A and B are connected graded k-algebras that are finitely generated in degree one, then $A \times_k B$ is finitely generated in degree one.*

Proof. Let $S = \{x_i\}_{i=1}^r \subseteq A_1$ and $T = \{y_i\}_{i=1}^s \subseteq B_1$ be generators. Take a homogenous element $a \otimes b \in A_d \otimes_k B_d$. We can write

$$a = \sum_{i=1}^m \alpha_i X_1^{(i)} \cdots X_d^{(i)} \text{ and } b = \sum_{j=1}^n \beta_j Y_1^{(j)} \cdots Y_d^{(j)}$$

for $\alpha_i, \beta_j \in k$, $(X_1^{(i)}, \ldots, X_d^{(i)}) \in \prod_{i=1}^d S$, and $(Y_1^{(j)}, \ldots, Y_d^{(j)}) \in \prod_{i=1}^d T$. Hence we have

$$\begin{aligned} a \otimes b &= \left(\sum_{i=1}^m \alpha_i X_1^{(i)} \cdots X_d^{(i)}\right) \otimes \left(\sum_{j=1}^n \beta_j Y_1^{(j)} \cdots Y_d^{(j)}\right) \\ &= \sum_{i=1}^m \left(\alpha_i X_1^{(i)} \cdots X_d^{(i)} \otimes \left(\sum_{j=1}^n \beta_j Y_1^{(j)} \cdots Y_d^{(j)}\right)\right) \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n \left(\alpha_i X_1^{(i)} \cdots X_d^{(i)} \otimes \beta_j Y_1^{(j)} \cdots Y_d^{(j)}\right)\right) \\ &= \sum_{i,j} \alpha_i \beta_j (X_1^{(i)} \otimes Y_1^{(j)}) \cdots (X_d^{(i)} \otimes Y_d^{(j)}) \end{aligned}$$

Therefore $A \times_k B$ is finitely generated in degree one by $\{x_i \otimes y_j\}_{i,j}$. □

As a nice corollary, we can relax the conditions on Van Rompay (1996, Theorem 2.4) to avoid the Noetherian conditions on the Segre and tensor products.

Theorem 3.6.3 (Van Rompay (1996, Theorem 2.4))**.** *Let A and B be finitely generated, connected graded k-algebras, and let $S = A \times_k B$, $T = A \otimes_k B$. If A and B are both generated in degree one, then there is an equivalence of categories*

$$\begin{aligned} \mathbb{V}\colon\ \mathrm{QGr}\, S &\longrightarrow \mathrm{QGr}\, T \\ E &\longmapsto \pi_T\left(T \otimes_S \omega_S E\right) \end{aligned}$$

Proof. As noted in Van Rompay's comments preceding the Theorem, the Noetherian hypothesis is necessary only to ensure that $\mathrm{QGr}\, S$ and $\mathrm{QGr}\, T$ are well-defined. Thanks to Proposition 3.2.3 and Lemma 3.4.4, the equivalence follows by running the same argument. □

Chapter 4

Graded Morita Theory: A Warmup

This section demonstrates how the tools of dg-categories yield a nice perspective on derived graded Morita. Compare with the well-known graded Morita statement in Zhang (1996).

In order to utilize the machinery of dg-categories, we must first translate chain complexes of graded modules into dg-categories. While one can naïvely regard this category as a dg-category by way of an enriched Hom entirely analogous to the ungraded situation, the relevant statements of Toën (2007) are better suited to the perspective of functor categories. As such, we adapt the association of a ringoid with one object to a ring from Section 2.2 to the graded situation, considering instead a ringoid with multiple objects.

4.1 Preliminaries on Ringoids and their Modules

Though these results are stated in fuller generality, in the sequel we will generally be concerned only with the groups $\mathbb{Z}$ and $\mathbb{Z}^2$. We begin our adaptation with our notion of ringoids with multiple objects.

Definition 4.1.1. To a G-graded k-algebra, A, associate the category $\mathcal{A}$ with objects the group G, morphisms given by

$$\mathcal{A}(g_1, g_2) = A_{g_2-g_1},$$

and composition defined by the multiplication $A_{g_2-g_1}A_{g_3-g_2} \subseteq A_{g_3-g_1}$.

The category $\mathcal{A}$ is naturally enriched over $\operatorname{Mod} k$. However, since we wish to deal with chain complexes, we will upgrade our enriching category to the category of chain complexes by viewing modules as chain complexes concentrated in degree zero. In particular, we regard $\mathcal{A}$ as a dg-category by considering the k-module of morphisms as the complex

$$\mathcal{A}(g_1, g_2)^n = \begin{cases} A_{g_2 - g_1} & \text{if } n = 0, \\ 0 & \text{else.} \end{cases}$$

with zero differential. From this point on, whenever we speak of modules, we will mean the full subcategory of the functor category $\operatorname{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}(k))$ consisting of $C(k)$-enriched functors, which we denote by $\operatorname{dgMod}(\mathcal{A})$.

As an unfortunate side effect of considering chain complexes of graded modules, there will be many instances where there are two simultaneous gradings on an object: homological degree and homogenous degree. We avoid the latter term, preferring weight, and use degree solely when referring to homological degree.

For clarity, consider the example of a complex of G-graded left A-modules, M. The degree n piece of M is the G-graded left A-module M^n. The weight g piece of the graded module M^n is the A_0-module of homogenous elements of (graded) degree g, M^n_g. Note that in this terminology, the usual morphisms of graded modules are the weight zero morphisms.

As mentioned above, we have a natural enrichment of the category of chain complexes of graded modules over a graded ring.

Definition 4.1.2. Denote by $\mathcal{C}(\operatorname{Gr} A)$ the dg-category with objects chain complexes of G-graded left A-modules and morphisms defined as follows.

We say that a morphism $f\colon M \to N$ of degree p is a collection of morphisms

$$f^n\colon M^n \to N^{n+p}$$

of weight zero. We denote by $\mathcal{C}\,(\mathrm{Gr}\,A)\,(M,N)^p$ the collection of all such morphisms, which we equip with the differential

$$d(f) = d_N \circ f + (-1)^{p+1} f \circ d_M$$

and define $\mathcal{C}\,(\mathrm{Gr}\,A)\,(M,N)$ to be the resulting chain complex. Composition is the usual composition of graded morphisms.

We denote by $\mathcal{C}\,(\mathrm{Gr}\,(A^{\mathrm{op}}))$ the same construction with G-graded right A-modules, which are equivalently left modules over the opposite ring, A^{op}.

Remark 4.1.3. One should note that the closed morphisms are precisely the morphisms of complexes $M \to N[p]$ and, in particular, the closed degree zero morphisms are precisely the usual morphisms of complexes.

The following lemma illustrates that modules and chain complexes are one and the same.

Lemma 4.1.4. *Let G be an abelian group. If A is a G-graded algebra over k and $\mathcal{A}$ the associated dg-category, then there is an isomorphism of dg-categories*

$$\mathcal{C}\,(\mathrm{Gr}\,A) \cong \mathrm{dgMod}\,(\mathcal{A})\,.$$

Proof. We first construct a dg-functor $F\colon \mathcal{C}\,(\mathrm{Gr}\,A) \to \mathrm{dgMod}\,(\mathcal{A})$. For each element g of G, denote by $A(g)[0]$ the complex with $A(g)$ in degree zero and consider the full subcategory of $\mathcal{C}\,(\mathrm{Gr}\,A)$ of all such complexes. We see that a morphism

$$f \in \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)^n$$

is just the data of a morphism $f^0\colon A(g) \to M^n$ which gives

$$\mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)^n \cong \mathrm{Gr}\,A(A(g), M^n) \cong M^n_{-g}$$

and hence $M_{-g} := \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)$ is the complex with M^n_{-g} in degree n. In particular, when $M = A(h)[0]$, we have

$$\mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], A(h)[0]) := A(h)[0]_{-g} = \mathcal{A}(g,h),$$

which allows us to identify this subcategory with $\mathcal{A}$ via the enriched Yoneda embedding, $A(h)[0]$ corresponding to the representable functor $\mathcal{A}(-,h)$. Using this identification, we can define the image of M in $\mathrm{dgMod}(\mathcal{A})$ to be the dg-functor that takes an object $g \in G$ to

$$M_{-g} = \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)$$

with structure morphism

$$\mathcal{A}(g,h) \cong \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], A(h)[0]) \to \mathcal{C}\,(k)\,(M_{-h}, M_{-g})$$

induced by the representable functor $\mathcal{C}\,(\mathrm{Gr}\,A)\,(-, M)$. We define the image of a morphism $f \in \mathcal{C}\,(\mathrm{Gr}\,A)\,(M,N)$ to be the natural transformation given by the collection of morphisms

$$h^{A(-g)[0]}(f)\colon \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(-g)[0], M) \to \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(-g)[0], N)$$

indexed by G.

Conversely, we note that the data of a functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}\,(k)$ is a collection of chain complexes, $M_g := M(g)$, indexed by G and morphisms of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & A_{g-h} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{C}\,(k)\,(M_g, M_h)^0 \arrow[r] & \mathcal{C}\,(k)\,(M_g, M_h)^1 \arrow[r] & \cdots
\end{tikzcd}$$

The non-zero arrow factors through $Z^0(\mathcal{C}\,(k)\,(M_g, M_h))$, so the structure morphism is equivalent to giving a morphism

$$A_{g-h} \to \mathcal{C}\,(k)\,(M_g, M_h)$$

and thus M determines a complex of graded A-modules

$$\widetilde{M} = \bigoplus_{g \in G} M_{-g}.$$

A morphism $\eta\colon M \to N$ is simply a collection of natural transformations η^p such that for each $g \in G$ we have $\eta^p(g) \in \mathcal{C}(k)(M_g, N_g)^p$ and the naturality implies that $\eta^p(g)$ is A-linear. The natural transformation η^p thus determines a morphism

$$\bigoplus_{g\in G} \eta^p(-g) \in \mathcal{C}(\operatorname{Gr} A)\left(\widetilde{M}, \widetilde{N}\right)^p,$$

and hence η determines a morphism in $\mathcal{C}(\operatorname{Gr} A)\left(\widetilde{M}, \widetilde{N}\right)$, which is the collection of all such homogenous components. This defines a dg-functor $\operatorname{dgMod}(\mathcal{A}) \to \mathcal{C}(\operatorname{Gr} A)$ which is clearly the inverse of F. $\square$

Remark 4.1.5. It is worth noting that it is natural from the ringoid perspective to reverse the weighting on the opposite ring in that, formally,

$$A_g^{\mathrm{op}} = \mathcal{A}^{\mathrm{op}}(0, g) = \mathcal{A}(g, 0) = A_{-g}$$

so that $\mathcal{A}^{\mathrm{op}}(-, h) = \mathcal{A}(h, -)$ is the representable functor corresponding to the left module $A^{\mathrm{op}}(h)$ by

$$\bigoplus_{g\in G} \mathcal{A}^{\mathrm{op}}(-g, h) = \bigoplus_{g\in G} \mathcal{A}(h, -g) = \bigoplus_{g\in G} A_{-(g+h)} = \bigoplus_{g\in G} A_{g+h}^{\mathrm{op}} = A^{\mathrm{op}}(h).$$

With this convention, when considering right modules, one can dispense with the formality of the opposite ring by constructing from a complex, M, the dg-functor $\mathcal{A} \to \mathcal{C}(k)$ mapping g to $M_g := \mathcal{C}(\operatorname{Gr}(A^{\mathrm{op}}))(A(-g)[0], M)$.

When $G = \mathbb{Z}^2$, and A, B are $\mathbb{Z}$-graded algebras over k, we denote the dg-category of chain complexes of G-graded B-A-bimodules by $\mathcal{C}(\operatorname{Gr} A^{\mathrm{op}} \otimes_k B)$. We associate to the $\mathbb{Z}^2$-graded k-algebra $A^{\mathrm{op}} \otimes_k B$ the tensor product of the associated dg-categories, $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}$. Note that, as in the remark above, in the identification

$$\mathcal{C}(\operatorname{Gr}(A^{\mathrm{op}} \otimes_k B)) \cong \operatorname{dgMod}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$$

the weighting coming from the A-module structure is reversed. The representable functors in this case are

$$\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}((-, -), (u, v)) := \mathcal{A}^{\mathrm{op}}(-, u) \otimes_k \mathcal{B}(-, v)$$

and correspond to $(A^{\mathrm{op}} \otimes_k B)(u,v) := A^{\mathrm{op}}(u) \otimes_k B(v)$ by

$$\bigoplus_{(x,y)\in\mathbb{Z}^2} \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\left((-x,-y),(u,v)\right) = \bigoplus_{(x,y)\in\mathbb{Z}^2} A^{\mathrm{op}}_{u+x} \otimes_k B_{v+y} = A^{\mathrm{op}}(u) \otimes_k B(v)$$

Remark 4.1.6. It is sometimes convenient to note the following. Let P be a chain complex of bi-bi A-modules. If $\mathcal{P}(m,n) = P_{m,-n}$ is the corresponding dg $\mathcal{A}$-$\mathcal{A}$-bimodule, then by the construction of the tensor product, it's easy to see that for any u the tensor product

$$\mathcal{A}(-,u) \otimes_{\mathcal{A}} \mathcal{P} \cong \mathcal{P}(u,-)$$

corresponds to the chain complex of left A-modules

$$\bigoplus_{n\in\mathbb{Z}} \mathcal{P}(u,-n) = \bigoplus_{n\in\mathbb{Z}} P_{u,n} = P_{u,*}$$

We will often identify P with $\mathcal{P}$, as well as $\mathcal{A}(-,u)$ with $A(u)$, and, under this identification, write $P \otimes_{\mathcal{A}} A(u) = P_{u,*}$.

Similarly, for any v, if we regard $A(v)$ as a right A-module, we will often write $A(v) \otimes_{\mathcal{A}} P = P_{*,-v}$ for the chain complex of right A-modules. We remark that as an artifact of the reverse weighting, we can homogenize these formulas by thinking of P as a left A^{op}-module, make the formal identification $A(-v) = A^{\mathrm{op}}(v)$ and then

$$A^{\mathrm{op}}(v) \otimes_{\mathcal{A}} P = A(-v) \otimes_{\mathcal{A}} P = P_{*,v}.$$

4.2 Derived Graded Morita Theory

From this construction, we have a dg-enhancement, $\text{h-proj}\,(\mathcal{A})$, of the derived category of graded modules, $\mathrm{D}(\mathrm{Gr}\,A)$. Passing through the machinery of Corollary 2.6.5, we have an isomorphism in $\mathrm{Ho}\,(\mathrm{dgcat}_k)$

$$\mathbf{R}\underline{\mathrm{Hom}}_c\left(\text{h-proj}\,(\mathcal{A})\,,\text{h-proj}\,(\mathcal{B})\right) \cong \text{h-proj}\,(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})\,,$$

so we identify an object, F, of $\mathbf{R}\underline{\mathrm{Hom}}_c\left(\text{h-proj}\,(A)\,,\text{h-proj}\,(B)\right)$ as a dg $\mathcal{A}$-$\mathcal{B}$-bimodule, P, which in turn corresponds to a morphism $\Phi_P\colon \mathcal{A} \to \text{h-proj}\,(\mathcal{B})$ by way of the symmetric monoidal closed structure on dgcat_k.

Following Section 3.3 of Canonaco and Stellari (2015), we identify the homotopy equivalence class, $[P]_{\mathrm{Iso}}$, of P with $[\Phi_P] \in [\mathcal{A}, \text{h-proj}(\mathcal{B})]$. The extension of Φ_P,

$$P \otimes_{\mathcal{A}} - = \widehat{\Phi_P}\colon \text{ h-proj}(\mathcal{A}) \to \text{h-proj}(\mathcal{B})$$

descends to a morphism $[\widehat{\Phi_P}] \in [\text{h-proj}(\mathcal{A}), \text{h-proj}(\mathcal{B})]$ and induces a triangulated functor that commutes with coproducts

$$\begin{aligned} H^0(\widehat{\Phi_P})\colon \mathrm{D}(\mathrm{Gr}\, A) &\longrightarrow \mathrm{D}(\mathrm{Gr}\, B) \\ M &\longmapsto P \otimes^{\mathbf{L}}_{\mathcal{A}} M. \end{aligned}$$

In particular, given an equivalence $f\colon \mathrm{D}(\mathrm{Gr}\, A) \to \mathrm{D}(\mathrm{Gr}\, B)$, we obtain from Lunts and Orlov (2010) a quasi-equivalence

$$F\colon \text{ h-proj}(\mathcal{A}) \to \text{h-proj}(\mathcal{B})\,.$$

Tracing through the remarks above, we obtain an object P of $\text{h-proj}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ providing an equivalence

$$H^0(\widehat{\Phi_P})\colon \mathrm{D}(\mathrm{Gr}\, A) \to \mathrm{D}(\mathrm{Gr}\, B).$$

Chapter 5

Derived Morita Theory for Noncommutative Projective Schemes

Let A and B be left Noetherian connected graded k-algebras. We want to extend the ideas from Chapter 4 to cover dg-enhancements of $\mathrm{D}(\mathrm{QGr}\, A)$.

5.1 Vanishing of a tensor product

We recall a particularly nice type of property of objects in the setting of compactly generated triangulated categories. In the sequel, many of our properties will be of this type, so we give this little gem a name.

Definition 5.1.1. Let $\mathcal{D}$ be a compactly generated triangulated category. Let $\mathtt{P}$ be a property of objects of $\mathcal{D}$. We say that $\mathtt{P}$ is **RTJ** if it satisifies the following three conditions.

- Whenever $A \to B \to C$ is a triangle in $\mathcal{D}$ and $\mathtt{P}$ holds for A and B, then $\mathtt{P}$ holds for C.
- If $\mathtt{P}$ holds for A, then $\mathtt{P}$ holds for the translate $A[1]$.
- Let I be a set and A_i be objects of $\mathcal{D}$ for each $i \in I$. If $\mathtt{P}$ holds for each A_i, then $\mathtt{P}$ holds for $\bigoplus_{i\in I} A_i$.

Proposition 5.1.2. *Let $\mathtt{P}$ be an RTJ property that holds for a set of compact generators of $\mathcal{D}$. Then $\mathtt{P}$ holds for all objects of $\mathcal{D}$.*

Proof. Let P be the full triangulated subcategory of objects for which $\mathtt{P}$ holds. Then P

- contains a set of compact generators,
- is triangulated, and
- is closed under formation of coproducts.

Thus, P is all of $\mathcal{D}$. $\square$

Definition 5.1.3. Let M be a complex of left graded A-modules and let N be a complex of right graded A-modules. We say that the pair satisfies $\bigstar(M, N)$ if we have vanishing of the tensor product

$$\mathbf{R}\tau_{A^{\mathrm{op}}} N \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{R}Q_A M = 0.$$

If $\bigstar(M, N)$ holds for all M and N, then we say that A satisfies $\bigstar$.

Proposition 5.1.4. *Let A be a finitely generated, connected graded k-algebra. Assume that $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_{A^{\mathrm{op}}}$ commute with coproducts. Then A satisfies $\bigstar$ if and only if $\bigstar(A(u), A(v))$ holds for each $u, v \in \mathbb{Z}$.*

Proof. The necessity is clear, so assume that $\bigstar(A(u), A(v))$ holds for each $u, v \in \mathbb{Z}$. First, we consider the property $\bigstar(M, A(v))$ of objects, M, of $\mathrm{D}(\mathrm{Gr}\, A)$. It's clear that this is an RTJ property that holds, by assumption, for the set of compact generators, $\{A(u)\}_{u \in \mathbb{Z}}$. Hence $\bigstar(M, A(v))$ holds for all M by Proposition 5.1.2.

Now fix any object M of $\mathrm{D}(\mathrm{Gr}\, A)$ and consider the property $\bigstar(M, N)$ of objects, N, of $\mathrm{D}(\mathrm{Gr}\, A^{\mathrm{op}})$. This is again an RTJ property for which $\bigstar(M, A(v))$ holds for all $v \in \mathbb{Z}$. By Proposition 5.1.2, $\bigstar(M, N)$ holds for all N . Since the choice of M was arbitrary, it follows that $\bigstar(M, N)$ holds for all M and for all N. Therefore A satisfies $\bigstar$. $\square$

There are various types of projection formulas. We record here two which will be useful in the sequel.

Proposition 5.1.5. *Let A be a finitely generated, connected graded k-algebra. Let P be a complex of bi-bi A-modules and let M be a complex of left graded A-modules. Assume $\mathbf{R}\tau_A$ commutes with coproducts. There is a natural quasi-isomorphism*

$$(\mathbf{R}\tau_A P) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \to \mathbf{R}\tau_A \left(P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \right).$$

Assume $\mathbf{R}Q_A$ commutes with coproducts. There is a natural quasi-isomorphism

$$(\mathbf{R}Q_A P) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \to \mathbf{R}Q_A \left(P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \right).$$

Proof. We treat the τ projection formula. The Q projection formula is analogous. By Corollary 3.5.7, we see that the tensor product is well-defined. It suffices to exhibit a natural transformation for the underived functors applied to modules to generate the desired natural transformation. Given

$$\psi \otimes_{\mathcal{A}} m \in \underline{\mathrm{Gr}}A(A/A_{\geq m}, P) \otimes_{\mathcal{A}} M$$

we naturally get

$$\begin{aligned} \widetilde{\psi} : A/A_{\geq m} &\to P \otimes_{\mathcal{A}} M \\ a &\mapsto \psi(a) \otimes_{\mathcal{A}} m. \end{aligned}$$

Taking the colimit gives the natural transformation.

Let us look at the natural transformation when $P = A(u) \otimes_k A(v)$, and $M = A(w)$. Recall from Remark 4.1.6 that

$$\mathbf{R}\tau_A(P) \otimes^{\mathbf{L}}_{\mathcal{A}} A(w) \cong \mathbf{R}\tau_A(P) \otimes_{\mathcal{A}} A(w) \cong \mathbf{R}\tau_A(P)_{*,w} := \bigoplus_{x \in \mathbb{Z}} \mathbf{R}\tau_A(P_{*,w})_x = \mathbf{R}\tau_A(P_{*,w})$$

which is compatible with the natural transformation. The property that the natural transformation is a quasi-isomorphism is RTJ in each entry. Thus, it holds for all P and M by Proposition 5.1.2. $\square$

For the hypothesis, recall Definition 3.5.12.

Proposition 5.1.6. *Assume A is delightful. Then $\bigstar$ holds for A.*

Proof. By Proposition 5.1.4, it suffices to check $\bigstar(M, A(v))$ for each v. This is equivalent to $\bigstar(M, \bigoplus_v A(v))$. Equipping the sum with a bi-bi structure as Δ, we reduce to checking $\bigstar(M, \Delta)$. Using Proposition 3.5.11 and Lemma 3.4.7 for A and A^{op}, we have a natural quasi-isomorphism

$$\mathbf{R}\tau_{A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M \cong \mathbf{R}\tau_A \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M.$$

Using Proposition 5.1.5, we have a natural quasi-isomorphism

$$\mathbf{R}\tau_A \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M \cong \mathbf{R}\tau_A \left(\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M\right) \cong \mathbf{R}\tau_A \left(\mathbf{R}Q_A M\right) = 0.$$

□

5.2 Duality

One can regard the bimodule $\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta$ as a sum of A-modules

$$\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta = \bigoplus_x (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,x}$$

and define for any object, M, of $\mathcal{C}\,(\mathrm{Gr}\,A)$ the object

$$\mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta) = \bigoplus_x \mathbf{R}\,\mathrm{Hom}_A(M, (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,x})$$

of $\mathcal{C}\,(\mathrm{Gr}\,(A^{\mathrm{op}}))$. Consider the functor

$$(-)^{\vee} : \mathcal{C}\,(\mathrm{Gr}\,A)^{\mathrm{op}} \to \mathcal{C}\,(\mathrm{Gr}\,(A^{\mathrm{op}}))$$
$$M \mapsto \mathbf{R}\underline{\mathrm{Hom}}_A\,(M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)$$

Lemma 5.2.1. *Assume A is delightful. Then the natural map*

$$\mathrm{id} \to (-)^{\vee\vee}$$

given by evaluation is a quasi-isomorphism for $\mathbf{R}Q_A A(x)$, for all x. Furthermore, there are quasi-isomorphisms

$$(\mathbf{R}Q_A A(x))^\vee \cong \mathbf{R}Q_{A^{\mathrm{op}}} A(-x).$$

Proof. We first exhibit the latter quasi-isomorphisms. Using the quasi-isomorphisms of Proposition 3.5.11, we obtain two quasi-isomorphic decompositions of $\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta$ as a sum of A-modules

$$(\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,j} \cong (\mathbf{R}Q_A\Delta)_{*,j} = \bigoplus_i (\mathbf{R}Q_A\Delta)_{i,j} = \bigoplus_i \mathbf{R}Q_A(\Delta_{*,j})_i = \mathbf{R}Q_A A(j)$$

and as a sum of A^{op}-modules

$$(\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{i,*} \cong (\mathbf{R}Q_{A^{\mathrm{op}}}\Delta)_{i,*} = \bigoplus_j (\mathbf{R}Q_{A^{\mathrm{op}}}\Delta)_{i,j} = \bigoplus_j \mathbf{R}Q_{A^{\mathrm{op}}}(\Delta_{i,*})_j = \mathbf{R}Q_{A^{\mathrm{op}}} A(i).$$

The first implies that $(\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,j}$ is right orthogonal to τ_A-torsion, hence by applying $\mathbf{R}\underline{\mathrm{Hom}}_A(-, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)$ to the triangle

$$\mathbf{R}\tau_A A(x) \to A(x) \to \mathbf{R}Q_A A(x)$$

we obtain a triangle

$$(\mathbf{R}\tau_A A(x))^\vee \cong 0 \to A(x)^\vee \xrightarrow{\sim} (\mathbf{R}Q_A A(x))^\vee.$$

Moreover, since $A(x)$ is compact, we also obtain a quasi-isomorphism

$$\begin{aligned}(\mathbf{R}Q_A A(x))^\vee \cong A(x)^\vee &= \bigoplus_j \mathbf{R}\operatorname{Hom}_A(A(x), (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,j}) \\ \cong \mathbf{R}\operatorname{Hom}_A\left(A(x), \bigoplus_j (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,j}\right) &= \mathbf{R}\operatorname{Hom}_A(A(x), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta) \\ &\cong (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{-x,*}\end{aligned}$$

and the second decomposition yields

$$(\mathbf{R}Q_A A(x))^\vee \cong (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{-x,*} \cong \mathbf{R}Q_{A^{\mathrm{op}}} A(-x).$$

Applying this twice, we get

$$(\mathbf{R}Q_A A(x))^{\vee\vee} \cong \mathbf{R}Q_A A(x).$$

We need only check that the natural map $\nu : 1 \to (-)^{\vee\vee}$ induces the identity after this quasi-isomorphism.

Note that we found a map

$$A(-x) \to \mathbf{R}Q_{A^{\mathrm{op}}} A(-x) \to (\mathbf{R}Q_A A(x))^{\vee}$$

inducing the quasi-isomorphism $(\mathbf{R}Q_A A(x))^{\vee\vee} \cong \mathbf{R}Q_A A(x)$. If α is the image of 1 in $\mathbf{R}Q_{A^{\mathrm{op}}} A(-x)$, denote by $\alpha^{\vee}$ the image in $(\mathbf{R}Q_A A(x))^{\vee}$. Since $1 \in A(-x)_x$, one can identify $\alpha^{\vee}$ as a morphism

$$\mathbf{R}Q_A A(x) \to (\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta)_{*,x} \cong \mathbf{R}Q_A A(x)$$

which is the natural inclusion. For any $a \in \mathbf{R}Q_A A(x)$ we obtain a morphism

$$\mathrm{ev}_a \colon (\mathbf{R}Q_A A(x))^{\vee} \to \mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta$$

and hence

$$\mathrm{ev}_a(\alpha^{\vee}) = \alpha^{\vee}(a) = a$$

Thus, we see that ν is quasi-fully faithful on $\mathbf{R}Q_A A(x)$ for all x. $\square$

Definition 5.2.2. Let $Q\mathcal{A}$ be the full dg-subcategory of $\mathcal{C}\,(\mathrm{Gr}\, A)$ with objects given by Q_A applied to injective resolutions of $A(x)$ for all x.

Corollary 5.2.3. *Assume that A is delightful. The functor $(-)^{\vee}$ induces a quasi-equivalence $(Q\mathcal{A})^{\mathrm{op}} \cong Q(\mathcal{A}^{\mathrm{op}})$.*

Proof. From Lemma 5.2.1, we see that $(-)^{\vee}$ is quasi-fully faithful on $Q\mathcal{A}$ and has quasi-essential image $Q(\mathcal{A}^{\mathrm{op}})$. $\square$

Lemma 5.2.4. *Assume that A is delightful. There is a natural map*

$$\eta : M^{\vee} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \operatorname{Hom}_A(M, N),$$

which is a quasi-isomorphism for all M and all $N \cong \mathbf{R}Q_A N$.

Proof. First, note that we have the natural map

$$M^{\vee} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \mathbf{R}\underline{\operatorname{Hom}}_A(M, \mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N).$$

For $M = A(x)$, we see this map is a quasi-isomorphism using the fact that A satisfies ★ from Proposition 5.1.6. Since A satisfies ★, the map

$$N \cong \Delta \otimes_{\mathcal{A}} N \to \mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N$$

is a quasi-isomorphism. So the map

$$\mathbf{R}\underline{\operatorname{Hom}}_A(M, N) \to \mathbf{R}\underline{\operatorname{Hom}}_A(M, \mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N)$$

is also a quasi-isomorphism. Combining the two gives the desired quasi-isomorphism for $M = A(x)$. But the condition η is a quasi-isomorphism is RTJ in M so is true for all M by Proposition 5.1.2 □

5.3 Products

Definition 5.3.1. For a graded k-algebra, A, let h-inj $(\operatorname{Gr} A)$ be the full dg-subcategory of $\mathcal{C}(\operatorname{Gr} A)$ with objects the K-injective complexes of Spaltenstein (1988). Similarly, we let h-inj $(\operatorname{QGr} A)$ be the full dg-subcategory of $\mathcal{C}(\operatorname{QGr} A)$ with objects the K-injective complexes.

Lemma 5.3.2. *The functor*

$$\omega : \text{h-inj}\,(\operatorname{QGr} A) \to \text{h-inj}\,(\operatorname{Gr} A)$$

is well-defined. Moreover, $H^0(\omega)$ is an equivalence with its essential image.

Proof. For the first statement, we just need to check that ω takes K-injective complexes to K-injective complexes. This is clear from the fact that ω is right adjoint to π, which is exact.

To see this is fully faithful, we recall that $\pi\omega \cong \mathrm{Id}$ so

$$\text{h-inj}\,(\mathrm{Gr}\,A)\,(\omega M, \omega N) \cong \text{h-inj}\,(\mathrm{QGr}\,A)\,(\pi\omega M, N) \cong \text{h-inj}\,(\mathrm{QGr}\,A)\,(M, N).$$

$\square$

Remark 5.3.3. Using Lemma 5.3.2, we can either use h-inj $(\mathrm{QGr}\,A)$ or its image under ω in h-inj $(\mathrm{Gr}\,A)$ as an enhancement of $\mathrm{D}(\mathrm{QGr}\,A)$.

Consider the full dg-subcategory of h-inj $(\mathrm{QGr}\,A \otimes_k B)$ consisting of the objects

$$\pi_{A \otimes_k B}(A(u) \otimes_k B(v))$$

for all u, v. Denote this subcategory by $\mathcal{E}$.

Lemma 5.3.4. *If A and B are both Ext-finite, left Noetherian, and right Noetherian, then the dg-category $\mathcal{E}$ is naturally quasi-equivalent to $Q\mathcal{A} \otimes_k Q\mathcal{B}$.*

Proof. Recall that $Q\mathcal{A}$ is the full dg-subcategory of $\mathcal{C}\,(\mathrm{Gr}\,A)$ consisting of Q_A applied to injective resolutions of $A(u)$, loosely denoted by $\mathbf{R}Q_A A(u)$, and similarly for $Q\mathcal{B}$. We have the exact functor

$$- \otimes_k - : \mathcal{C}\,(\mathrm{Gr}\,A) \otimes_k \mathcal{C}\,(\mathrm{Gr}\,B) \to \mathcal{C}\,(\mathrm{Gr}\,A \otimes_k B)$$

which tensors a pair of modules over k to yield a bimodule. First consider the triangle

$$\begin{aligned} \mathbf{R}\tau_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)) &\to \mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v) \\ &\to \mathbf{R}Q_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)). \end{aligned}$$

By Proposition 3.5.9, we have

$$\mathbf{R}Q_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)) \cong \mathbf{R}Q_A\left(\mathbf{R}Q_B\left(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)\right)\right).$$

Since $\mathbf{R}\tau_B$ commutes with coproducts, we have a natural quasi-isomorphism

$$\begin{aligned}\mathbf{R}Q_A\left(\mathbf{R}Q_B\left(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)\right)\right) &\cong \mathbf{R}Q_A^2 A(u) \otimes_k \mathbf{R}Q_B^2 B(v) \\ &\cong \mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v).\end{aligned}$$

Thus,

$$\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v) \to \mathbf{R}Q_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v))$$

is a quasi-isomorphism for all u, v with $\tau_{A\otimes_k B}$ torsion cone. The same consideration shows that the map

$$A(u) \otimes_k A(v) \to \mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)$$

induces a quasi-isomorphism

$$\mathbf{R}Q_{A\otimes B}(A(u) \otimes_k B(v)) \to \mathbf{R}Q_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v))$$

with $\tau_{A\otimes_k B}$ torsion kernel. Now we check that these morphisms induce quasi-isomorphisms on the morphism spaces giving our desired quasi-equivalence. We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hom}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v), \mathbf{R}Q_A A(x) \otimes_k \mathbf{R}Q_B B(y)) \arrow[dr, "c"] \arrow[ddl, "a"'] & \\
 & & \mathrm{Hom}(A(u) \otimes_k B(v), \mathbf{R}Q_A A(x) \otimes_k \mathbf{R}Q_B B(y)) \arrow[dd, "d"] \\
\mathrm{Hom}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v), \mathbf{R}Q_{A\otimes_k B}(\mathbf{R}Q_A A(x) \otimes_k \mathbf{R}Q_B B(y))) \arrow[drr, "e"] & & \\
 & & \mathrm{Hom}(A(u) \otimes_k B(v), \mathbf{R}Q_{A\otimes_k B}(\mathbf{R}Q_A A(x) \otimes_k \mathbf{R}Q_B B(y))) \\
\mathrm{Hom}(\mathbf{R}Q_{A\otimes_k B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)), \mathbf{R}Q_{A\otimes_k B}(\mathbf{R}Q_A A(x) \otimes_k \mathbf{R}Q_B B(y))) \arrow[uu, "b"] & &
\end{tikzcd}$$

and we want to know first that a and b are quasi-isomorphisms. We know that b is a quasi-isomorphism since $\mathbf{R}\tau_{A\otimes_k B}$ is left orthogonal to $\mathbf{R}Q_{A\otimes_k B}$ so we only need to check a. Since $A(u) \otimes_k B(v)$ is free and

$$\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v) \to \mathbf{R}Q_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v))$$

is a quasi-isomorphism, d is a quasi-isomorphism. Since $\mathbf{R}Q_A$ and $\mathbf{R}Q_B$ commute with coproducts, using tensor-Hom adjunction shows that c is a quasi-isomorphism. Finally, since the cone over the map

$$A(u) \otimes_k A(v) \to \mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v)$$

is annihilated by $\tau_{A \otimes_k B}$, we see that e is also a quasi-isomorphism. This implies that a is a quasi-isomorphism. By an analogous argument, the endomorphisms of $\mathbf{R}Q_{A\otimes B}(A(u) \otimes_k B(v))$ and $\mathbf{R}Q_{A\otimes B}(\mathbf{R}Q_A A(u) \otimes_k \mathbf{R}Q_B B(v))$ are quasi-isomorphic. $\square$

5.4 The quasi-equivalence

We now turn to the main result.

Theorem 5.4.1. *Let k be a field. Let A and B be connected graded k-algebras. If A and B form a delightful couple, then there is a natural quasi-equivalence*

$$F : \text{h-inj}\left(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B\right) \to \mathbf{R}\underline{\mathrm{Hom}}_c\left(\text{h-inj}\left(\mathrm{QGr}\, A\right), \text{h-inj}\left(\mathrm{QGr}\, B\right)\right)$$

such that for an object P of $\mathrm{D}(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$, the exact functor $H^0(F(P))$ is isomorphic to

$$\Phi_P(M) := \pi_B\left(\mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}\omega_A M\right).$$

Proof. Applying Corollary 2.6.5, it suffices to provide a quasi-equivalence

$$G : \text{h-inj}\left(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B\right) \to \text{h-proj}\left((Q\mathcal{A})^{\mathrm{op}} \otimes_k Q\mathcal{B}\right).$$

Using Corollary 5.2.3, we have a quasi-equivalence

$$\text{h-proj}\left((Q\mathcal{A})^{\mathrm{op}} \otimes_k Q\mathcal{B}\right) \cong \text{h-proj}\left(Q\mathcal{A}^{\mathrm{op}} \otimes_k Q\mathcal{B}\right).$$

From Lemma 5.3.4 we have a quasi-fully faithful functor

$$\imath \colon Q\mathcal{A}^{\mathrm{op}} \otimes_k Q\mathcal{B} \to \text{h-inj}\left(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B\right)$$

which induces a dg-functor

$$\imath^*\colon \text{h-inj}\,(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B) \to \mathrm{dgMod}\,(Q\mathcal{A}^{\mathrm{op}} \otimes_k Q\mathcal{B})$$

mapping an object P of h-inj $(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$ to the dg-functor

$$\begin{aligned} (Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B})^{\mathrm{op}} &\longrightarrow \mathcal{C}\,(k) \\ E &\longmapsto \text{h-inj}\,(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)\,(\imath E, P). \end{aligned}$$

We first note that, because the image of objects of $Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B}$ are compact objects of h-inj $(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$, for any set J the natural map

$$\imath^*\left(\bigoplus_{j\in J} P_j\right) \to \bigoplus_{j\in J} \imath^*(P_j)$$

is a quasi-isomorphism, so $\imath^*$ is continuous. By making the identification of the object $P_{u,v} = \pi_{A^{\mathrm{op}}\otimes_k B}(A(u) \otimes_k B(v))$ with an object of dgMod $(Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B})$, we obtain the quasi-isomorphism

$$\imath^*(P_{u,v}) = \text{h-inj}\,(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)\,(\imath(-), P_{u,v}) \cong \mathrm{dgMod}\,(Q\mathcal{A}^{\mathrm{op}} \otimes_k Q\mathcal{B})\,(-, P_{u,v})$$

and, consequently, the quasi-isomorphism

$$\begin{aligned} \mathrm{dgMod}\,(Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B})\,(\imath^*(P_{u,v}), \imath^*(P_{u',v'})) &\cong \imath^*(P_{u',v'})(P_{u,v}) \\ &= \text{h-inj}\,(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)\,(P_{u,v}, P_{u',v'}). \end{aligned}$$

Since the collections $\{\imath^* P_{u,v}\}_{\mathbb{Z}^2}$ and $\{P_{u,v}\}_{\mathbb{Z}^2}$ are a set of compact generators for h-proj $(Q\mathcal{A}^{\mathrm{op}} \otimes_k Q\mathcal{B})$ and h-inj $(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$, respectively, it follows that $\imath^*$ is a quasi-equivalence between h-inj $(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$ and h-proj $(Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B})$, the full dg-subcategory of compact objects of dgMod $(Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B})$, by Proposition 2.7.3.

Tracing out the quasi-equivalences, one just needs to manipulate

$$\mathrm{Hom}(\mathbf{R}Q_A A(x)^\vee \otimes_k \mathbf{R}Q_B B(y), P) \cong$$

$$\mathrm{Hom}(\mathbf{R}Q_B B(y), \mathrm{Hom}(\mathbf{R}Q_A A(x)^\vee, \mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P)) \cong$$

$$\mathrm{Hom}(\mathbf{R}Q_B B(y), \mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A A(x))$$

using Propostion 5.1.6 and Lemma 5.2.4. This says that the induced continuous functor is

$$M \mapsto \pi_B \left(\mathbf{R}\omega_{A^{\mathrm{op}} \otimes_k B} P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}\omega_A M \right).$$

□

The following statement is now a simple application of Theorem 5.4.1 and results of Lunts and Orlov (2010).

Corollary 5.4.2. *Let A and B be a delightful couple of connected graded k-algebras with k a field. Assume that there exists an equivalence*

$$f : \mathrm{D}(\mathrm{QGr}\, A) \to \mathrm{D}(\mathrm{QGr}\, B).$$

Then there exists an object $P \in D(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$ such that

$$\Phi_P : \mathrm{D}(\mathrm{QGr}\, A) \to \mathrm{D}(\mathrm{QGr}\, B)$$

is an equivalence.

Proof. Applying Lunts and Orlov (2010, Theorem 1) we know there is a quasi-equivalence between the unique enhancements, that is a morphism

$$F : \text{h-inj}\,(\mathrm{QGr}\, A) \to \text{h-inj}\,(\mathrm{QGr}\, B)$$

of $\mathrm{Ho}\,(\mathrm{dgcat}_k)$ inducing an equivalence

$$H^0(F) : H^0(\text{h-inj}\,(\mathrm{QGr}\, A)) = \mathrm{D}(\mathrm{QGr}\, A) \to H^0(\text{h-inj}\,(\mathrm{QGr}\, B)) = \mathrm{D}(\mathrm{QGr}\, B).$$

By Theorem 5.4.1, there exists a $P \in \mathrm{D}(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$ such that $\Phi_P = H^0(F)$. □

We wish to identify the kernels as objects of the derived category of an honest noncommutative projective scheme. In general, one can only hope that kernels obtained as above are objects of the derived category of a noncommutative (bi)projective scheme. However, we have the following special case in which we can collapse the $\mathbb{Z}^2$-grading to a $\mathbb{Z}$-grading.

Corollary 5.4.3. *Let A and B be a delightful couple of connected graded k-algebras with k a field that are both generated in degree one. Assume that there exists an equivalence*

$$f : \mathrm{D}(\mathrm{QGr}\, A) \to \mathrm{D}(\mathrm{QGr}\, B).$$

Then there exists an object $P \in D(\mathrm{QGr}\, A^{\mathrm{op}} \times_k B)$ that induces an equivalence

$$\begin{aligned} \mathrm{D}(\mathrm{QGr}\, A) &\longrightarrow \mathrm{D}(\mathrm{QGr}\, B) \\ M &\longmapsto \pi_B \left(\mathbb{V}_{dg}(P) \otimes^{\mathbf{L}} \mathbf{R}\omega_A M \right) \end{aligned}$$

Proof. The equivalence $\mathbb{V}$ of Theorem 3.6.3 extends naturally to a quasi-equivalence

$$\mathbb{V}_{\mathrm{dg}} \colon \text{ h-inj}\,(\mathrm{QGr}\, S) \to \text{h-inj}\,(\mathrm{QGr}\, T)\,.$$

Now we can choose P such that $\mathbb{V}_{\mathrm{dg}}(P)$ is homotopy equivalent to the kernel obtained by applying Corollary 5.4.2, so the desired equivalence is $\Phi_{\mathbb{V}_{\mathrm{dg}}(P)}$. □

Bibliography

Anel, Mathieu and Bertrand Toën (2009). “Dénombrabilité des classes d’équivalences dérivées de variétés algébriques”. In: *J. Algebraic Geom.* 18.2, pp. 257–277. ISSN: 1056-3911. DOI: 10.1090/S1056-3911-08-00487-6.

Arcara, Daniele et al. (2013). “The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability”. In: *Adv. Math.* 235, pp. 580–626. ISSN: 0001-8708. DOI: 10.1016/j.aim.2012.11.018.

Artin, M. (1997). “Some problems on three-dimensional graded domains”. In: *Representation theory and algebraic geometry (Waltham, MA, 1995)*. Vol. 238. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, pp. 1–19.

Artin, M. and William F. Schelter (1987). “Graded algebras of global dimension 3”. In: *Adv. in Math.* 66.2, pp. 171–216. ISSN: 0001-8708. DOI: 10.1016/0001-8708(87)90034-X.

Artin, M. and J. T. Stafford (1995). “Noncommutative graded domains with quadratic growth”. In: *Invent. Math.* 122.2, pp. 231–276. ISSN: 0020-9910. DOI: 10.1007/BF01231444.

Artin, M., J. Tate, and M. Van den Bergh (1990). “Some algebras associated to automorphisms of elliptic curves”. In: *The Grothendieck Festschrift, Vol. I*. Vol. 86. Progr. Math. Birkhäuser Boston, Boston, MA, pp. 33–85.

Artin, M. and J. J. Zhang (1994). “Noncommutative projective schemes”. In: *Adv. Math.* 109.2, pp. 228–287. ISSN: 0001-8708. DOI: 10.1006/aima.1994.1087.

Ballard, Matthew, David Favero, and Ludmil Katzarkov (2014). “A category of kernels for equivariant factorizations and its implications for Hodge theory”. In: *Publ. Math. Inst. Hautes Études Sci.* 120, pp. 1–111. ISSN: 0073-8301. DOI: 10.1007/s10240-013-0059-9.

Bayer, Arend and Emanuele Macrì (2014a). “MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations”. In: *Invent. Math.* 198.3, pp. 505–590. ISSN: 0020-9910. DOI: 10.1007/s00222-014-0501-8.

Bayer, Arend and Emanuele Macrì (2014b). "Projectivity and birational geometry of Bridgeland moduli spaces". In: *J. Amer. Math. Soc.* 27.3, pp. 707–752. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-2014-00790-6.

Ben-Zvi, David, John Francis, and David Nadler (2010). "Integral transforms and Drinfeld centers in derived algebraic geometry". In: *J. Amer. Math. Soc.* 23.4, pp. 909–966. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-10-00669-7.

Bondal, A. and D. Orlov (June 1995). "Semiorthogonal decomposition for algebraic varieties". In: *eprint arXiv:alg-geom/9506012*.

Bondal, A. and M. Van den Bergh (2003). "Generators and representability of functors in commutative and noncommutative geometry". In: *Mosc. Math. J.* 3.1, pp. 1–36, 258. ISSN: 1609-3321.

Borceux, Francis (1994). *Handbook of categorical algebra. 2*. Vol. 51. Encyclopedia of Mathematics and its Applications. Categories and structures. Cambridge University Press, Cambridge, pp. xviii+443. ISBN: 0-521-44179-X.

Bridgeland, Tom (2002). "Flops and derived categories". In: *Invent. Math.* 147.3, pp. 613–632. ISSN: 0020-9910. DOI: 10.1007/s002220100185.

— (2006). "Derived categories of coherent sheaves". In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, pp. 563–582.

— (2007). "Stability conditions on triangulated categories". In: *Ann. of Math. (2)* 166.2, pp. 317–345. ISSN: 0003-486X. DOI: 10.4007/annals.2007.166.317.

Candelas, Philip et al. (1991). "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory". In: *Nuclear Phys. B* 359.1, pp. 21–74. ISSN: 0550-3213. DOI: 10.1016/0550-3213(91)90292-6.

Canonaco, Alberto and Paolo Stellari (2015). "Internal Homs via extensions of dg functors". In: *Adv. Math.* 277, pp. 100–123. ISSN: 0001-8708. DOI: 10.1016/j.aim.2015.03.004.

Drinfeld, Vladimir (2004). "DG quotients of DG categories". In: *J. Algebra* 272.2, pp. 643–691. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2003.05.001.

Dyckerhoff, Tobias (2011). "Compact generators in categories of matrix factorizations". In: *Duke Math. J.* 159.2, pp. 223–274. ISSN: 0012-7094. DOI: 10.1215/00127094-1415869.

Gabriel, Pierre (1962). "Des catégories abéliennes". In: *Bull. Soc. Math. France* 90, pp. 323–448. ISSN: 0037-9484.

Goodearl, K. R. and R. B. Warfield Jr. (2004). *An introduction to noncommutative Noetherian rings*. Second. Vol. 61. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, pp. xxiv+344. ISBN: 0-521-83687-5; 0-521-54537-4. DOI: 10.1017/CBO9780511841699.

Hovey, Mark (1999). *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, pp. xii+209. ISBN: 0-8218-1359-5.

— (2001). "Model category structures on chain complexes of sheaves". In: *Trans. Amer. Math. Soc.* 353.6, pp. 2441–2457. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-01-02721-0.

Huybrechts, D. (2006). *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, pp. viii+307. ISBN: 978-0-19-929686-6; 0-19-929686-3. DOI: 10.1093/acprof:oso/9780199296866.001.0001.

Kawamata, Yujiro (2002). "D-equivalence and K-equivalence". In: *J. Differential Geom.* 61.1, pp. 147–171. ISSN: 0022-040X.

Keller, Bernhard (1994). "Deriving DG categories". In: *Ann. Sci. École Norm. Sup. (4)* 27.1, pp. 63–102. ISSN: 0012-9593.

— (2006). "On differential graded categories". In: *International Congress of Mathematicians. Vol. II.* Eur. Math. Soc., Zürich, pp. 151–190.

Kontsevich, Maxim (1995). "Homological algebra of mirror symmetry". In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*. Birkhäuser, Basel, pp. 120–139.

Lesieutre, John (2015). "Derived-equivalent rational threefolds". In: *Int. Math. Res. Not. IMRN* 15, pp. 6011–6020. ISSN: 1073-7928. DOI: 10.1093/imrn/rnu117.

Lunts, Valery A. and D. Orlov (2010). "Uniqueness of enhancement for triangulated categories". In: *J. Amer. Math. Soc.* 23.3, pp. 853–908. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-10-00664-8.

Mukai, Shigeru (1981). "Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves". In: *Nagoya Math. J.* 81, pp. 153–175. ISSN: 0027-7630.

— (1987). "On the moduli space of bundles on $K3$ surfaces. I". In: *Vector bundles on algebraic varieties (Bombay, 1984)*. Vol. 11. Tata Inst. Fund. Res. Stud. Math. Tata Inst. Fund. Res., Bombay, pp. 341–413.

Orlov, D. (1997). "Equivalences of derived categories and $K3$ surfaces". In: *J. Math. Sci. (New York)* 84.5. Algebraic geometry, 7, pp. 1361–1381. ISSN: 1072-3374. DOI: 10.1007/BF02399195.

Polishchuk, Alexander and Arkady Vaintrob (2012). "Chern characters and Hirzebruch - Riemann - Roch formula for matrix factorizations". In: *Duke Math. J.* 161.10, pp. 1863–1926. ISSN: 0012-7094. DOI: 10.1215/00127094-1645540.

Reiten, I. and M. Van den Bergh (2002). "Noetherian hereditary abelian categories satisfying Serre duality". In: *J. Amer. Math. Soc.* 15.2, pp. 295–366. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-02-00387-9.

Schwede, Stefan and Brooke Shipley (2003). "Stable model categories are categories of modules". In: *Topology* 42.1, pp. 103–153. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(02)00006-X.

Spaltenstein, N. (1988). "Resolutions of unbounded complexes". In: *Compositio Math.* 65.2, pp. 121–154. ISSN: 0010-437X.

Stafford, J. T. (2002). "Noncommutative projective geometry". In: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*. Higher Ed. Press, Beijing, pp. 93–103.

Stenström, Bo (1975). *Rings of quotients. An introduction to methods of ring theory.* Vol. 217. Die Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, New York - Heidelberg, pp. viii+309.

Stephenson, D. R. (1996). "Artin-Schelter regular algebras of global dimension three". In: *J. Algebra* 183.1, pp. 55–73. ISSN: 0021-8693. DOI: 10.1006/jabr.1996.0207.

— (1997). "Algebras associated to elliptic curves". In: *Trans. Amer. Math. Soc.* 349.6, pp. 2317–2340. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-97-01769-8.

Tabuada, Goncalo (2005). "Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories". In: *C. R. Math. Acad. Sci. Paris* 340.1, pp. 15–19. ISSN: 1631-073X. DOI: 10.1016/j.crma.2004.11.007.

Toën, Bertrand (2007). "The homotopy theory of *dg*-categories and derived Morita theory". In: *Invent. Math.* 167.3, pp. 615–667. ISSN: 0020-9910. DOI: 10.1007/s00222-006-0025-y.

Van den Bergh, M. (2001). "Blowing up of non-commutative smooth surfaces". In: *Mem. Amer. Math. Soc.* 154.734, pp. x+140. ISSN: 0065-9266. DOI: 10.1090/memo/0734.

Van Rompay, Kristel (1996). "Segre product of Artin-Schelter regular algebras of dimension 2 and embeddings in quantum $\mathbf{P}^3$'s". In: *J. Algebra* 180.2, pp. 483–512. ISSN: 0021-8693. DOI: 10.1006/jabr.1996.0078.

Verdier, Jean-Louis (1996). "Des catégories dérivées des catégories abéliennes". In: *Astérisque* 239. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis, xii+253 pp. (1997). ISSN: 0303-1179.

Weibel, Charles A. (1994). *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1.

Zhang, J. J. (1996). "Twisted graded algebras and equivalences of graded categories". In: *Proc. London Math. Soc. (3)* 72.2, pp. 281–311. ISSN: 0024-6115.